Recursive Utility with Unbounded Aggregators\textsuperscript{1}

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Abstract. A new framework is presented for the study of the existence and uniqueness of solutions to the Koopmans equation in the unbounded case, that is based on the concept of $k$–local contraction mapping. For this purpose, suitable metrics are introduced in the space of weak* continuous functions defined on subspaces of positive real sequences. In this way, there is no need to consider a limited class of functions, as happens with the weighted norm approach. The results are applied to the study of some of the more remarkable examples in the literature, and also to undiscounted and even upcounted models.

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1 Introduction

Recursive utility is used in many contexts to model economic agents’ preferences where current utility is expressed as a function (the aggregator) of current consumption and the utility of future consumption.

Traditionally, the functional relation between current utility and future utility has been considered as one additive and separable over time. This assumption has two major implications. On the one hand, for any initial date, the preference order on consumption streams after this period is independent of the behavior of the agent on the initial time interval. On the other hand, fixed consumption streams from any time onwards do not influence the preference order on the initial time interval.

The assumption of additive separability has been removed in recent years, mainly because of their limited application and rigidity in the modelling of agents’ preferences. However, it has some important features, such as analytical tractability and recursive properties, allowing dynamic programming methods to solve the consumption maximization problem.

A major criticism of the additive separability hypothesis comes from the undesirable fact that the marginal substitution rate of current for future utility is constant and therefore independent of utility levels. This may yield some extreme results, such as in models with heterogeneous consumers, where the most patient household ends up with all the capital in the long run, while the others consume nothing, as shown in [1]. This counterintuitive result is avoided, allowing the marginal rate of substitution to depend on consumption. The recursive utility model allows this rate to be variable while implying a weak separability between present and future consumption. This fact is sufficient to apply stationary dynamic programming techniques.

A more severe drawback of the additive separable utility occurs in a risk environment. Here, as well as the aforementioned marginal rate of substitution, a very important
concept is the risk aversion of the agents. However, with additive separability, it is impossible to disentangle these two quite different attributes of consumers’ tastes (see [7, 9]). Risk aversion concerns the agents’ attitude to the variation in consumption across states of the world at a fixed date. The inflexibility of expected additive utility makes the isolation of both effects, risk aversion and marginal rate of substitution, impossible.

The study of recursive utility was pioneered by Koopmans in [10], who first characterized the utility function by means of an aggregator associated to time–stationary preferences. In [12] is taken the converse way, considering the aggregator function as primitive, and then determining the associated recursive utility. Their approach is based on contraction mapping arguments and limited to bounded aggregators, obtaining a unique bounded recursive utility function. Their techniques do not allow the unbounded aggregator case to be covered.

Among the contributions with regard to the unbounded case we must cite the important contributions [2, 3, 5], which focus on a different technique, also appealing to contraction mapping arguments. The approach taken by these authors is based on the introduction of a weighted norm on a certain space of continuous functions obeying a growth condition. Some assumptions linking this growth rate and the discounting rate permit them to assure the existence of a unique recursive utility in the prescribed class of functions. Although their treatment covers the unbounded case, and it is, to our knowledge, one of the most useful and general approach in existence, it seems to be difficult to implement in a general context because it needs to find the adequate weighted function. In our approach this requirement is substituted by another more mechanical and, in our opinion, more natural one. Furthermore, for the case of bounded consumption streams the class of functions under consideration need not be restricted, thus obtaining the uniqueness of the recursive utility with respect to the whole class of continuous functions. Another important byproduct of our framework is that the recursive utility
function is in fact continuous with respect to the weak* topology, and not only with respect to the weighted norm topology, as obtained in [3] or [5]. This fact has important implications from the point of view of the myopia of the agents. Myopia is related with the notion of impatience and, from a technical point of view, with the continuity of the utility function. Continuity with respect to some topologies implies that present consumption is preferred to future consumption. An extreme case is the product topology, where continuity means that the utility function is not sensitive to long term variations in consumption. Other topologies can also be related to impatience, as suggested in [4] with the Mackey topology. In [18] is showed that the fixed point of the Koopmans operator is Mackey continuous when it is defined on the set of bounded consumption streams $\ell^\infty$. In fact, as proven in Theorems 3 and 4, the recursive utility function inherits a stronger property, this being weak* continuity. Weak* topology is the weaker Hausdorff locally convex topology such that the topological dual of $\ell^\infty$ is $\ell^1$, the set of summable sequences. In [4] is proved that Mackey topology is the strongest Hausdorff locally convex topology on $\ell^\infty$ that is strongly myopic, so it is clear that the weak* topology is also strongly myopic (refer to [3] for an excellent exposition of these concepts).

Contraction techniques are not the only way to tackle the problem. In [5] a partial sum method is introduced for the so-called "unbounded below problem", where the aggregator becomes $-\infty$ at some points. Basically, the approach consists of studying the following limit:

$$\lim_{t \to \infty} W(c_1, W(c_2, \ldots, W(c_t, y) \ldots)),$$

where $W$ is the aggregator function, $c_t$ represents the consumption in period $t$ and $y$ is a constant. Monotonicity of $W$ in both variables join some other hypotheses – related with the weighted contraction techniques used in [5] in the bounded below case – assure the existence and uniqueness of the recursive utility function, which is upper
semicontinuous and can take the value $-\infty$ at some points\(^1\). The limit above must be the recursive utility function $U$. But $U$ also verifies

$$U(c) = \lim_{t \to \infty} W(c_1, W(c_2, \ldots, W(c_t, U(\sigma^{t+1}c)) \ldots)),$$

where $c = (c_1, \ldots, c_t, \ldots)$, $\sigma$ is the shift operator and $\sigma^t$ is the $t$--fold composition of $\sigma$. Comparing both limits, it must be the case that the influence of $\sigma^t c$ on $U$ will become unimportant as $t \to \infty$, leading to the conclusion that $U$ exhibits time impatience. Roughly speaking, this property establishes a necessary assumption in the form of a Lipschitz condition for the aggregator with respect to tomorrow utility. In fact, if $\beta$ is the Lipschitz constant, then $U$ must satisfy $\lim_{t \to \infty} \beta^t U(\sigma^{t+1}(c)) = 0$, completely analogous to the transversality condition appearing in the context of dynamic programming. The transversality condition makes the importance of having a rate of impatience bounded by one clear.

Another interesting approach is developed in [16, 16] by introducing the notions of lower and upper convergence, leading to the concept of biconvergence, which also applies to dynamic programming models. This makes sense for an order interval of consumption streams. However, biconvergence implies a joint restriction on the technology, impatience, and preferences.

We introduce a different approach to the problem, although it rely on metric properties and on the Banach Contraction Theorem. However, normed spaces are not considered. We consider metric spaces of functions whose metric is constructed from a countable family of seminorms. The family is defined as the supremum of continuous functions over a family of compact sets covering the set of consumption streams. The freedom in the choice of the family makes this approach more flexible than others. Furthermore, the metric can be obtained in different ways. When the consumption streams are bounded,

\[^1\text{This approach is further developed in C. Le Van, Y. Vailakis, Recursive utility and optimal growth with bounded or unbounded returns, unpublished manuscript (2003).}\]
we choose a metric defined on the whole space of continuous functions—with respect to the weak* topology—thus obtaining a unique recursive utility function. This result is new in the literature. On the other hand, when a more general behavior of the consumption profiles is admitted, we define a different metric on a more limited set of functions, exploiting the properties of the Koopmans operator. These properties are denominated in this paper "0–local contraction" and "1–local contraction" and both become natural with the standard assumptions on the aggregator, that is to say, continuity in both variables and Lipschitz continuity with respect to the second variable. Monotonicity with respect to this variable is not needed.

It is worth noting that continuity, and not only upper semicontinuity, is obtained for the recursive utility. On the other hand, our treatment of the problem can handled aggregators unbounded below, since the problem of extending the utility to more general domains can be solved with the partial summation techniques, giving rise to an upper semicontinuous utility function.

The paper is organized as follows. Section 2 contains some preliminary definitions and results. Section 3 presents the main features of the recursive utility function based on the aggregator approach and establishes the framework chosen to study the problem. Section 4 is devoted to the statement of the results of the existence and uniqueness of the recursive utility function and to show the applicability of those results to the more remarkable examples in the literature. The proofs are in the Appendix.

2 $k$–local contraction mappings

In this section some definitions and the main results that will be used in the study of the Koopmans equation are introduced. The methodological approach followed was established by the authors in [14] in the context of dynamic programming. In that paper,
the specificity of the problem at hand led the authors to consider the topological space of continuous functions. However, the results can be extended, at no cost, to more general topological spaces sharing some of the characteristics of the space of continuous functions.

Let $X$ be some space such that there is a countable family of semidistances $\{d_j\}$ defined on it such that $d_j(x, y) = 0$ for all $j \in \mathbb{N}$ implies $x = y$. In this situation, a metric $d$ can be defined on $X$ in terms of the sequence $\{d_j\}$ as follows

$$d(x, y) = \sum_{j=1}^{\infty} 2^{-j} \frac{d_j(x, y)}{1 + d_j(x, y)}.$$  (1)

Another useful metric $d_c$ can also be defined on subsets $A$ of $X$ in terms of the family $\{d_j\}$, given positive constants $c$ and $M$, and a fixed element $x_0 \in X$. To do so, let us define $A$ as the subset of elements $x \in X$ satisfying

$$d_c(x, x_0) = \sum_{j=1}^{\infty} c^j d_j(x, x_0) \leq M,$$  (2)

and note that the metric $d_c$ is well defined on $A$. We suppose that both $(X, d)$ and $(A, d_c)$ are complete metric spaces.

We shall now define two local contraction concepts\(^2\) for operators defined on $X$. The first is 0–Local Contraction, which requires the operator to be a contraction from each of the semidistances $d_j$ and the second, 1–Local Contraction, asks the operator to link $d_j$ with $d_{j+1}$ as a contraction.

**Definition 1** Let $T : X \to X$ be an operator.

(i) $T$ is said to be a 0–Local Contraction (0–LC) if and only if

$$d_j(Tx, Ty) \leq \beta_j d_j(x, y)$$

for all $j \in \mathbb{N}$ and for all $x, y \in X$, where $0 \leq \beta_j < 1$.

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\(^2\)As we have already noted, these concepts were established by the authors in [14] in the context of dynamic programming.
(ii) \( T \) is said to be a 1–Local Contraction (1–LC) if and only if
\[
d_j(Tx, Ty) \leq \beta d_{j+1}(x, y)
\]
for all \( j \in \mathbb{N} \) and for all \( x, y \in X \), where \( 0 \leq \beta < 1 \).

Of course, the previous definitions also make sense on any subset \( A \) of \( X \). The role played by these concepts is stated in the two following theorems, which will be the main tools that yield the existence and uniqueness of the recursive utility in a fairly general context.

**Theorem 1.** Let \( T : X \to X \) be a 0–LC. Then, for every \( x_0 \in X \), the operator \( T \) maps the closed and bounded subset
\[
A = \left\{ x \in X : d_j(x, x_0) \leq \frac{d_j(Tx_0, x_0)}{1 - \beta_j} \quad \forall j \in \mathbb{N} \right\}
\]
into itself. Furthermore,

(a) \( T \) is a contraction on \( A \) and admits a fixed point \( x^* \) on \( A \), that is unique on \( X \).

(b) For any \( x_0 \in X \), \( T^nx_0 \xrightarrow{d} x^* \) as \( n \to \infty \).

**Theorem 2.** Let \( T : X \to X \) be a 1–LC. Then, for every \( x_0 \in X \) satisfying \( d_c(Tx_0, x_0) < \infty \) for some \( c > \beta \), the operator \( T \) maps the set
\[
A = \left\{ x \in X : d_c(x, x_0) \leq \frac{d_c(Tx_0, x_0)}{1 - \beta/c} \right\}
\]
into itself. Furthermore,

(a) \( T \) is a contraction on \( A \) and admits a unique fixed point \( x^* \) on \( A \).

(b) For any \( x \in A \), \( T^nx \xrightarrow{d} x^* \) as \( n \to \infty \).

\(^3\)The set \( A \) is closed and bounded with respect to the topology generated by the metric \( d \).
3 Recursive utility

Recursive utility appears in many economic contexts in which the agents’ preference order can be expressed as a function $W$, called the aggregator, depending on current consumption and the utility of future consumptions. The aggregator function $W$ maps $X \times Y$ to $Y$, where $X$ is a subset of $\mathbb{R}_+ = \{ x \in \mathbb{R} : x \geq 0 \}$ and $Y$ is a subset of $\mathbb{R}$. $W$ defines a recursive operator $\mathcal{K}$ on the space of real functions defined over the set of non-negative real sequences $\mathbb{R}^\mathbb{N}_+ = \{ \mathbf{c} = (c_1, c_2, \ldots) \in \mathbb{R}^\mathbb{N} : c_t \geq 0 \ \forall t \in \mathbb{N} \}$. When referring to real sequences in $\mathbb{R}^\mathbb{N}$, subscripts will denote components and superscripts will be used to distinguish different real sequences. Thus, a real sequence $\mathbf{c}$ will sometimes be denoted by means of its components, $\mathbf{c} = (c_t)$.

Let us define the projection $\pi$ and shift $\sigma$ by $\pi \mathbf{c} = c_1$ and $\sigma \mathbf{c} = (c_2, c_3, \ldots)$, respectively, for all $\mathbf{c} = (c_1, c_2, \ldots) \in \mathbb{R}^\mathbb{N}_+$. The operator $\mathcal{K}$ associated to $W$ is defined by $\mathcal{K}U(\mathbf{c}) = W(\pi \mathbf{c}, U(\sigma \mathbf{c}))$. If $\pi$ and $\sigma$ are continuous with respect to some topology $\tau$, then the operator $\mathcal{K}$ maps continuous functions on continuous functions –in the $\tau$ topology– provided the aggregator is a continuous real function.

A real function $U$ defined on a set $D \subseteq \mathbb{R}^\mathbb{N}_+$ is called recursive if it verifies the Koopmans Equation: $U(\mathbf{c}) = \mathcal{K}U(\mathbf{c}) = W(\pi \mathbf{c}, U(\sigma \mathbf{c}))$, that is to say, $U$ is a fixed point of $\mathcal{K}$ with fundamental $W$. The idea behind this is that $U$ can be found by means of recursive substitution of itself in the Koopmans Equation. The recursion makes sense on $D$ if $\pi(\bigcup_{t=0}^{\infty} \sigma^t D) \subseteq X$, where $\sigma^t \mathbf{c} = (c_{t+1}, c_{t+2}, \ldots)$ for all $t \in \mathbb{N}$. This property will be assumed throughout the paper. We denote for future reference $\pi^t \mathbf{c} = c_t$. We also say that $\mathbf{c} < \mathbf{c}'$ if $c_t < c'_t$ for all $t \in \mathbb{N}$, and $\mathbf{c} \leq \mathbf{c}'$ if $c_t \leq c'_t$ for all $t \in \mathbb{N}$.

The following hypotheses are standard in the literature (see [3] or [12]).

(W1) $W$ is continuous on $X \times Y$. 

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(W2) \( W \) obeys a Lipschitz condition with respect to \( y \)

\[
|W(x, y) - W(x, y')| \leq \beta(x) |y - y'|,
\]

for all \( x \in X \) and for all \( y, y' \in Y \), where \( \beta : X \rightarrow [0, \infty) \) is a continuous function.

In the seminal paper [12], the assumption of boundeness of the aggregator is imposed. One of the aims of this paper is to overcome this limitation.

A property is now imposed on commodity spaces that will be the main key in later results. The idea we have in mind is to consider commodity spaces so that it is possible to set a complete metric on the space of continuous functions defined on it.

**Definition 2** A commodity space \( D \) of \( \mathbb{R}^N_+ \) is called *admissible* if it is endowed with a topology \( \tau \) such that either of the following holds

(a) \( D \) is compact, or

(b) \( D = \bigcup_{j=1}^{\infty} K_j \) is the union of an increasing and countable family of compact subsets \( \{K_j\} \), such that for every compact \( K \subseteq D \), there exists \( j \in \mathbb{N} \) with \( K \subseteq K_j \).

The first requirement for \( D \) in (b) is called in the literature \( \sigma \)-compactness, whereas the second is called hemicompactness. For \( D \) admissible, let \( C(D) \) denote the set of real functions defined on \( D \) that are continuous with respect to the topology \( \tau \). We also occasionally use the symbol \( B(D) \) to denote the set of real locally bounded functions, that is, functions which are bounded on compact subsets of \( D \). Obviously \( C(D) \subseteq B(D) \).

We can define a countable family of semidistances (seminorms) \( \{p_j\} \) on \( C(D) \) and \( B(D) \) by setting

\[
p_j(U) = \sup\{|U(c)| : c \in K_j\} = \|U\|_{K_j} \quad \forall j = 1, 2, \ldots,
\]

where, it is obvious that \( p_j(U) = 0 \) for every \( j \) implies \( U \) is the null function. Now we can construct the metrics \( d \) and \( d_c \) as in the above section, with \( d_j = p_j \), via

\[
d(U, V) = \sum_{j=1}^{\infty} 2^{-j} \frac{\|U - V\|_{K_j}}{1 + \|U - V\|_{K_j}} \quad \text{and} \quad d_c(U, V) = \sum_{j=1}^{\infty} c^j \|U - V\|_{K_j},
\]
respectively, and the same construction is valid for functions in $B(D)$. It is straightforward to see that $(C(D),d) \ (B(D),d)$ and $(\mathbb{A},d_c)$, with $\mathbb{A}$ as defined in (2) are complete. One can adapt the proof of a similar result contained, e.g., in [15, pp31–32]). Hemicompactness is crucial to attain the result.

Thus, admissibility of the commodity space $D$ is the key for defining a complete metric on $C(D)$ or in some subsets of $C(D)$. Thus, throughout this paper, the set $D$ will be imposed to satisfy this property.

It is worth noting that convergence in distance $d$ means uniform convergence on the compact sets of $D$, and the same property is true for the metric $d_c$. In the next section we show that, under suitable conditions, the Koopmans operator is a contraction on $C(D)$ (on $\mathbb{A}$) under the metric $d\ (d_c)$.

Assumption (W1) assumes continuity for the aggregator $W$, so, for $K$ to map $C(D)$ into $C(D)$, it is sufficient to consider a topology $\tau$ with respect to which $\pi$ and $\sigma$ are continuous$^4$. In doing so, any continuous function will be transformed by the operator $K$ into a continuous function. Another important fact about the topology $\tau$ is that it must be selected in such a way that the consumption space $D$ is admissible in the sense of Definition 2. Hence, the selection of a topology $\tau$ is not a minor problem since it must be adequate to assure that i) the consumption space is admissible, and ii) the operators $\pi$ and $\sigma$ are both continuous. Note that property i) eliminates the product topology as a candidate for our purposes. To make this clear, consider the commodity space $D = \bigcup_j [0,j]_{\infty} = \bigcup_j K_j$ with the product topology, where $K_j = [0,j]_{\infty}$ is a compact set by the Tychonoff Theorem. Obviously, the sequence$^5 \{c^n\}$, with $c^n = (0, \ldots, 0, \frac{n}{n}, 0, \ldots) \in D$, converges to $0$ in the product topology. Consequently, $K = \{c^n\} \cup \{0\}$ is compact, although it is not contained in $K_j$ for any $j$. Thus, under the choice of the

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$^4$To do this, it is sufficient to consider some topology stronger than the product topology.

$^5$We can consider sequences instead of nets, since the product topology is metrizable. Thus in this situation the concepts of compact set and sequentially compact set coincide.
family of compact sets \( \{K_j\} \), the commodity space \( D \) is not admissible. The problem is not remedied with the selection of a more general family of compact sets. Consider an arbitrary family of compact sets \( \{K_j\} \) covering \( D \). Then \( \pi^t(K_j) \subseteq \mathbb{R}_+ \) is compact for all \( t \), so \( \alpha_{ij} = \max \pi^t(K_j) \) is well defined. It is obvious, however, that the compact set \( K = \{c^n\} \cup \{0\} \), where \( c^n = (0, \ldots, 0, 1 + \alpha_{mn}, 0, \ldots) \), is not contained in any \( K_j \).

The above comment leads us to consider the weak* topology\(^6\) defined on \( D \). The idea behind this selection is that, when \( D \) is assumed to be the norm dual of a separable Banach space, then Alaoglu’s Theorem states that any weak*-closed and norm–bounded subset of \( D \) is weak*-compact. In this situation, since \( D \) is a normed space, it can be covered by means of a countable family of increasing closed balls, which are in fact compacts from Alaoglu’s Theorem, so \( D \) is \( \sigma \)-compact (c.f. footnote 5). Now, as every weak*-compact subset is norm–bounded, any compact subset \( K \) of \( D \) will be contained in some compact of the family covering \( D \), which means that hemicompactness holds. Hence, \( D \) is an admissible commodity space in the sense of Definition 2. Note that we usually work on closed subsets of a norm dual space—e.g. the set of nonnegative and bounded real sequences—where the above remarks are also true.

We shall now consider different admissible commodity spaces, all with the weak* topology. Our aim is to show that most common and usual consumption sets are admissible under the weak* topology. Before doing so, it is important to note that, to apply Theorems 1 and 2, the way in which a commodity space \( D \) is covered by a family of compact sets \( \{K_j\} \) is conditioned by the necessity for the Koopmans operator to be a 0-LC or a 1-LC. As is shown in Theorems 3 and 4 below, these requirements are accomplished if the compact sets \( \{K_j\} \) covering \( D \) satisfy the conditions \( \sigma K_j \subseteq K_j \) or \( \sigma K_j \subseteq K_{j+1} \), respectively.

\(^6\)An alternative characterization of this topology to that given in the Introduction, valid for a Banach space \( E \), is as follows: the weak* topology on the norm dual \( E^* \) is the weakest topology such that \( E \) is the topological dual of \( E^* \).

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Commodity spaces are presented here as closed subsets of Riesz ideals. This procedure was introduced in [3, 5], where it is considered that commodity spaces are Riesz ideals, $A(w)$, generated by a single element $w$ of $\mathbb{R}^N$. Formally, for a given $w$, the set

$$A(w) = \{ c \in \mathbb{R}^N : |c| \leq \lambda w \text{ for some } \lambda > 0 \}$$

is the minimum Riesz ideal generated by $w$. It is straightforward to see that $A(w)$ is the norm dual of the separable Banach space of sequences $q \in \mathbb{R}^N$ satisfying $\|q\| = \sum_{t=1}^{\infty} |q_t| |w_t| < \infty$. Hence, in accordance with the comments above, $A(w)$ is an admissible commodity space under the weak* topology, so is any closed subset of $A(w)$. For instance, if $w = (1,1,\ldots)$, then the set of bounded sequences, $A(w) = \ell^N$, is an admissible commodity space. Similarly, if $w = (1,\alpha,\alpha^2,\ldots)$, with $\alpha > 0$, then the set of sequences with a growth rate bounded by $\alpha$, $A(w) = \ell^N(\alpha)$, is also admissible. Conveniently, most common commodity spaces used in economic analysis—namely $\ell^N_+, \ell^N_+(\alpha)$, and $\ell^N_+(\mu, \alpha)$ which is defined below—are then admissible, as they are closed subsets of a convenient norm dual space. In fact, for $w \in \mathbb{R}^N_+$, the non-negative cone

$$A_+(w) = \{ c \in \mathbb{R}^N_+ : c \leq \lambda w \text{ for some } \lambda > 0 \}$$

of the Riesz space $A(w)$ is also closed, and therefore admissible.

For general unbounded aggregator we cannot expect existence of an associated utility function on the whole $\mathbb{R}^N_+$ as it occurs in the bounded case studied in [12]. Instead we will consider consumption sequences which are bounded in a broad sense, allowing unbounded sequences showing a growth rate driven by some sequence $w$.

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For $\alpha > 0$ and $c \in \ell^N_+(\alpha)$, $\|c\|_\alpha = \sup_{t \in \mathbb{N}} \{|c_t|/\alpha^{t-1}\}$ defines a family of different norms in this set. This kind of weighted norms was introduced by [5]. A well known fact establishes that the norm unit ball of $\ell^N_+(\alpha)$ is not norm–compact. For this reason, and due to the fact that our approach rests on the assumption that the commodity space is covered by a countable family of compact sets, we have chosen the weak* topology, which is weaker than those derived from the above norms.
Let us now show how to cover $A_+(w)$ with a suitable family of compact sets, depending on $w$. First, take the case where the sequence $w$ is such that $A_+(w)$ coincides with the space of nonnegative bounded sequences $\ell^N_+$ (i.e. $w = (1, 1, \ldots)$). For this case, the selection of compact sets $K_j = [0, j]^{\infty}$ fulfills the conditions imposed in Definition 2 and furthermore, $\sigma K_j \subseteq K_j$ is satisfied.

When $w = (w_t)$ is an unbounded and increasing sequence, the family of compact sets $K_j' = \prod_{t=1}^{\infty} [0, w_{t+j-1}]$ is a sensible choice for covering $A_+(w)$, although in this case $\sigma K_j' \subseteq K_j'+1$ but not $\sigma K_j' \subseteq K_j'$. When the aggregator is not bounded below—typically $W(0, y) = -\infty$—we need to avoid consumption paths close to zero consumption. To do this, let $v = (v_t)$ and $w = (w_t)$ be two sequences in $\mathbb{R}^N_+$ such that $v$ is a decreasing sequence, $w$ is an increasing sequence, and $0 < v < 1 < w$. Then, $A_+(v, w)$ can be defined via

$$A_+(v, w) = \{ c \in \mathbb{R}^N_+ : \lambda_1 v \leq c \leq \lambda_2 w \text{ for some } \lambda_1, \lambda_2 > 0 \},$$

which is the set of paths that have a growth rate between $v$ and $w$. For this case the family of compact sets $K_j'' = \prod_{t=1}^{\infty} [v_{t+j-1}, w_{t+j-1}]$ constitutes a good selection, since it covers $A_+(v, w)$, and, as above, $\sigma K_j'' \subseteq K_j''+1$. It is worth noting that when $v = (1, \mu, \mu^2, \ldots)$ and $w = (1, \alpha, \alpha^2, \ldots)$, $A_+(v, w)$ coincides with

$$\ell^\infty_+ (\mu, \alpha) = \{ c \in \mathbb{R}^N_+ : \mu \| c \| > 0, \| c \|_\alpha < \infty \},$$

where $\mu \| c \| = \inf_{t \in \mathbb{N}} \{|c_i|/\mu^{t-1}\}$, $\| c \|_\alpha = \sup_{t \in \mathbb{N}} \{|c_i|/\alpha^{t-1}\}$, and $0 < \mu < 1 < \alpha$. As we have already noted, this kind of consumption sets was introduced by [5].

The compactness of the sets $K_j$, $K_j'$ and $K_j''$ we have used to cover the different domains is obvious since, in all cases, we have selected order intervals, which are compact under the weak* topology, from Alaoglu’s Theorem.

Another very useful set appears whenever the domain $D$ is just a compact set of $\mathbb{R}^N_+$. In such a case, the sequence of compact sets $\{K_j\}$ defined via $K_j = S^{j-1}(D)$ covers
proving that \( D \), and, due to this construction, satisfies \( S(K_j) = K_{j+1} \). Hence, since compactness is actually a stronger condition than admissibility, this kind of compact domains satisfies the required property considered in our framework, even though assumption (b) in Definition (2) does not hold. As we shall show later, these domains are particularly useful for tackling undiscounted and upcounted models.

4 Main results

Our first result in this section assures that \( K \) is a 0–LC on \( C(D) \), whenever \( D \) is an admissible commodity space satisfying \( \sigma K_j \subseteq K_j \). Therefore, the conclusions of Theorem 1 are applicable to the operator \( K \).

**Theorem 3** Let \( W \) be an aggregator satisfying (W1) and (W2). Let \( D \) be an admissible commodity space satisfying \( \sigma(K_j) \subseteq K_j \) and \( \beta_j = \max_{c \in K_j} \pi(c) < 1 \), for all \( j \in \mathbb{N} \). Then

(a) The Koopmans Equation has a unique solution \( U^* \) on \( C(D) \). Furthermore, \( U^* \) satisfies

\[
\|U^*\|_{K_j} \leq \frac{\|K_0\|_{K_j}}{1 - \beta_j} \quad \text{for all } j \in \mathbb{N}.
\]

(b) For every \( U \in C(D) \), \( K^n U \xrightarrow{d} U^* \) as \( n \to \infty \).

**Remark 1** (i) As explained in the previous section, the above theorem is also applicable when the more general class of locally bounded functions, \( B(D) \), is considered. In this case assumption (W1) can be substituted by imposing local boundedness on \( W \) with respect to the first variable.
(ii) Theorem 3 can be applicable, in some particular cases, when \( \sigma(K_j) \subseteq K_j \) does not hold. The trick consists in finding a closed and bounded subset \( A \) of continuous functions mapped into itself by the operator \( K \), and also that

\[
\max_{c \in \sigma K_j} |U(c) - V(c)| \leq \gamma_j \max_{c \in K_j} |U(c) - V(c)|, \quad \text{for all } U, V \in A,
\]

for some \( \{\gamma_j\} \) verifying \( \beta_j \gamma_j < 1 \) for all \( j \in \mathbb{N} \). Along the same lines of the proof of the theorem, \( K \) is a 0–LC with parameters \( \beta_j \gamma_j < 1 \). Thus there is a unique solution to the Koopmans Equation in \( A \). The above argument can be used for example in the homogeneous case which is explored in [6], assuming that both \( X \) and \( Y \) are cones and the aggregator satisfies \( W(\lambda x, \lambda^\theta y) = \lambda^\theta W(x, y) \) for all \( \lambda \geq 0 \) if \( \theta \geq 0 \) or \( \lambda > 0 \) if \( \theta < 0 \). Then he shows that the Koopmans operator maps homogeneous functions to homogeneous functions. If the degree of homogeneity is non negative, then it is easy to show the existence of a homogeneous solution whenever \( \beta_j \alpha^\theta < 1 \) and \( D = \ell_+^N(\alpha) \). In fact, this inequality implies that the Koopmans operator is a contraction on the set

\[
A = \{ U \in C(\ell_+^N(\alpha)) : U \text{ is homogeneous of degree } \theta \text{ and } \| U \|_{K_j} \leq m_j \ \forall \ j \in \mathbb{N} \},
\]

where \( m_j = \| K_0 \|_{K_j} / (1 - \beta_j \alpha^\theta) \). Due to the homogeneity of the function belonging to \( A \), it is very easy to verify that the inequality \( \max_{c \in \sigma K_j} |U(c) - V(c)| \leq \alpha^\theta \max_{c \in K_j} |U(c) - V(c)| \) still holds. Hence \( K \) is a 0–LC on \( A \) and, in consequence, a unique fixed point exists on \( A \), which is homogeneous of degree \( \theta \), and can be approached from every point of \( A \).

The case \( \theta < 0 \) will be analyzed later in Example 7.

(iii) We shall now consider the possibility of the existence of non–continuous fixed points arising as solutions to some particular problems of the Koopmans Equation. With this idea in mind, take \( D = \ell_+^N \) as the commodity space and consider the familiar aggregator \( W(x, y) = x + \beta y \), with \( \beta < 1 \). Then Theorem 3 implies that the Koopmans Equation has a unique solution on \( C(\ell_+^N) \), which is the function \( U^* (c) = \sum_{j=1}^\infty \beta^{j-1} c_j \). To locate a non–continuous fixed point \( V^* \) for \( K \) on \( \ell_+^N \), we proceed as follows. First, take any
non–constant path \( b \in \ell^N_+ \); for instance \( b = (b_1, b_2, \ldots) \) where \( b_i = 1/2^{i-1} \) for all \( i \in \mathbb{N} \).

Next, define \( V^* : \ell^N_+ \to \mathbb{R} \) as follows

\[
V^*(c) = \begin{cases} 
U^*(c_1, \ldots, c_p, 0) - \beta^{p-q} U^*(b_1, \ldots, b_q, 0) & \text{if } \sigma^p(c) = \sigma^q(b) \text{ for some } p, q \in \mathbb{N} \cup \{0\}, \\
U^*(c) & \text{otherwise.}
\end{cases}
\]

The verification that \( V^* \) is recursive is purely mechanical, so the details have been omitted. Now, since \( U^* \) is continuous on \( \ell^N_+ \), we have, by the definition of \( V^* \), that

\[
\lim_{t \to \infty} V^*(b_1, \ldots, b_t, 0) = \lim_{t \to \infty} U^*(b_1, \ldots, b_t, 0) = U^*(b).
\]

Then, as it is clear that \( V^*(b) = 0 \neq 2/(2 - \beta) = U^*(b) \), it follows that \( V^* \) is discontinuous\(^8\). Indeed, \( V^* \) is not locally bounded, as expected, because

\[
\lim_{t \to \infty} V^*(\sigma^t b) = \lim_{t \to \infty} -\beta^{-t} \sum_{j=1}^{t} \left( \frac{\beta}{2} \right)^{j-1} = -\infty.
\]

**Example 1 (Epstein–Hynes).** Consider the aggregator \( W(x, y) = (-1 + y)e^{-x} \), which is a special form of the aggregator \( W(x, y) = (-1 + y)e^{-v(x)} \) used in [8] for the case \( v(x) = x \).

Take \( D = A_+(v, 1) \), where \( v = (v_t) \) is any decreasing sequence of positive real numbers such that \( \lim_{t \to \infty} v_t = 0 \). For this case, the family of compact sets \( K_j = \prod_{t=1}^{\infty} [v_{t+j-1}, j] \) is a good choice since it covers \( A_+(v, 1) \) and satisfies \( \sigma(K_j) \subseteq K_j \).

Taking \( \beta(x) = e^{-x} \), (W2) holds. Given that \( \sup_{c \in K_j} \beta(\pi(c)) = e^{-v_j} = \beta_j < 1 \), Theorem 3 assures the existence of a unique continuous fixed point \( U^* \) on \( C(A_+(v, 1)) \).

In fact, this function is given by

\[
U^*(c) = -\sum_{t=1}^{\infty} e^{-(c_1+\cdots+c_t)},
\]

and can be calculated as \( \lim_{n \to \infty} K^n U \) for any \( U \in C(A_+(v, 1)) \).

---

\(^8\)Note that the sequence \((b_1, \ldots, b_n, 0)\) converges in the weak* topology to \( b \).
Example 2 (Decreasing rate of impatience). Take $a, b > 0$ with $a > b^2/4$ and define the aggregator $W : X \times Y \rightarrow \mathbb{R}$ as $W(x, y) = x + a + b\sqrt{y}$. Set $X = [0, \infty)$. Then, since $W \geq a$, $Y \subseteq [a, \infty]$, and consequently

$$\frac{\partial W}{\partial y} = \frac{b}{2\sqrt{y}} \leq \frac{b}{2\sqrt{a}} < 1.$$ 

It follows that this aggregator obeys a Lipschitz condition with respect to $y$ ($\beta(x) = \frac{b}{2\sqrt{a}}$), and then Theorem 3 assures that the Koopmans Equation has a unique solution $U^*$ on $C(\ell_+^\mathbb{N})$. Although it is no possible to find a closed-form representation for $U^*(c)$, it may be numerically approximated to any degree of accuracy by means of the sequence $K^n\theta(c)$, for any $\theta \in \mathbb{R}$.

Although the usefulness of Theorem 3 is evident, its requirements are quite restrictive for the consumption set. In particular, its application is restricted to bounded sequences of consumption or to specific problems as shown in Remark 1(ii). We shall now establish a more general existence result on admissible commodity spaces such as $A_+(w) \ (1 < w)$ or $A_+(v, w) \ (0 < v < 1 < w)$. In all these cases the property $\sigma(K_j) \subseteq K_{j+1}$ holds, instead of $\sigma(K_j) \subseteq K_j$.

The observation that under the aforementioned property, the Koopmans operator is a 1–LC with respect to the distance $d_c$, and therefore it has a fixed point, will allow us to extend the applicability of our approach. Remember that the distance $d_c$ is defined as

$$d_c(U, V) = \sum_{j=1}^{\infty} c^j \|U - V\|_{K_j},$$

where $U, V$ belong to a suitable subset of $C(D)$ in which the series converges.

The following result is the counterpart of Theorem 3 in this case.

Theorem 4. Let $W$ be an aggregator satisfying (W1) and (W2). Let $D$ be an admissible commodity space satisfying $\sigma(K_j) \subseteq K_{j+1}$ for all $j \in \mathbb{N}$, and let $U_0 \in C(D)$ such that
the series
\[ d_c(KU_0, U_0) = \sum_{j=1}^{\infty} c^j \| KU_0 - U_0 \|_{K_j} \]
converges for some \( c > 0 \) with \( \sup_{x \in X} \beta(x) \equiv \beta < c \). Then

(a) The Koopmans operator has a unique solution \( U^* \) on the set
\[ \mathbb{A} = \left\{ U \in C(D) : d_c(U, U_0) \leq \frac{d_c(KU_0, U_0)}{1 - \beta/c} \right\} \]

(b) For every \( U \in \mathbb{A} \), \( K^n U \xrightarrow{d_c} U^* \) as \( n \to \infty \).

**Remark 2.** In applications, it is important to determine the range of values of \( c \) for which the series \( \sum_{j=1}^{\infty} c^j \| KU_0 - U_0 \|_{K_j} \) converges. Here is a well known sufficient result: Let \( \sum_{j=1}^{\infty} c^j a_j \) be a power series of real (or complex) numbers and let \( \lambda \) and \( r \) be the real numbers defined by
\[ \lambda = \limsup_{j \to \infty} |a_j|^r \quad \text{and} \quad r = \frac{1}{\lambda}, \tag{5} \]
respectively (where \( r = 0 \) whenever \( \lambda = +\infty \) and \( r = +\infty \) whenever \( \lambda = 0 \)). If \( c < r \), then the series converges uniformly. If \( c > r \), then the series diverges.

Notice that in this context, where \( a_j = \| KU_0 - U_0 \|_{K_j} \), the limit given in (5) always exists (supposing we admit \( \infty \) as a possible value), since \( a_j \) is an increasing sequence of real numbers.

**Remark 3.** (i) It is easy to show that the hypotheses imposed in Boyd’s Theorem, [5], imply that \( K \) is a 1–LC. Let us suppose there exists a strictly positive function \( \varphi \), continuous with respect to some weighted norm \( \| \cdot \|_\alpha \), satisfying for every \( c \in \ell_+^M(\alpha) \),
\[ |K0(c)| \leq d \varphi(c) \]
where \( d > 0 \), and
\[ \beta \sup_{c \in \ell_+^M(\alpha)} \frac{\varphi(\sigma c)}{\varphi(c)} = \bar{d} < 1. \]
Theses are the hypotheses of Boyd’s Theorem. Since \( \ell N + (\alpha) \) can be covered by means of a sequence \( \{K_j\} \) satisfying \( \sigma K_j = K_{j+1} \) (c.f. \( K_j = \prod_{t=1}^{\infty}[0, \alpha^{-j+t-1}] \)), the latter inequality gives
\[
\|\varphi\|_{K_{j+1}} < \beta^{-1} \bar{d} \|\varphi\|_{K_j} < \cdots < \beta^{-j} \bar{d}^j \|\varphi\|_{K_1}.
\]
This and \( |\mathcal{K}(c)| \leq d \varphi(c) \) imply \( \|\mathcal{K}0\|_{K_{j+1}} \leq d \beta^{-j} \bar{d}^j \|\varphi\|_{K_1} \). Thus, in order to apply Theorem 4 it is then sufficient to consider \( c \) as satisfying \( \beta < c < \beta/\bar{d} \).

(ii) With the assumptions of the theorem fulfilled, what we obtain is that the Koopmans operator has a unique fixed point, which is unique on \( A \) and continuous in the weak* topology. Furthermore, the solution can be approached by successive iterations. In the unbounded below case, there remains the question of whether the solution can be extended from \( A_+(v, w) \) to the whole \( A_+(w) \). [5] gives conditions assuring the existence of an upper semicontinuous solution— in the topology generated by \( \| \cdot \|_\alpha \). Their technique of proof relies on partial summations that approach in the limit the desired recursive utility function. Actually, the hypotheses in the Upper Semicontinuous Existence Theorem of Boyd easily imply that \( \mathcal{K} \) is a 1–LC. The argument proving this fact is similar to those contained in item (i) of this remark, so it is omitted. Then, the conclusions of Theorem 4 can be strengthened in the sense that the upper semicontinuous solution defined on \( A_+(w) \) is in fact weak* continuous on \( A_+(v, w) \).

**Example 3 (Epstein-Hynes II).** Consider the aggregator \( W(x, y) = (-1 + y)e^{-v(x)} \) with \( v > 0 \), introduced in [8]. Notice first that (W2) holds for \( \beta = e^{-v(0)} < 1 \). now, take \( D = A_+(w) \), where \( w = (w_t) \) is any increasing and unbounded real sequence. Then, by using the null function as \( U_0 \), we obtain
\[
\limsup_{j \to \infty} \|\mathcal{K}0\|_{K_j}^{1/j} = \limsup_{j \to \infty} \|e^{-v(c_1)}\|_{K_j}^{1/j} = \lim_{j \to \infty} e^{-v(0)/j} = 1.
\]
Thus, Theorem 4 applies, yielding a recursive utility function \( U^* \) on \( C(A_+(w)) \) that is
given by

\[ U^*(c) = -\sum_{t=1}^{\infty} e^{-(v(c_1) + \cdots + v(c_t))}. \]

Moreover, since the limit given in (6) does not depend on the particular choice of \( w \), Theorem 4 assures that \( U^* \) is the unique recursive weak* continuous function defined on \( A_+(w) \), for any \( w \).

**Example 4 (Koopmans–Diamond–Williamson).** Let \( W(x, y) = \beta/d \ln (1 + ax^b + dy) \), where \( a, b, d > 0 \) and \( 0 < \beta < 1 \), introduced in [11]. Take any increasing and unbounded real sequence \( w = (w_t) \) satisfying \( \lim \sup_{j \to \infty} (\ln(1 + aw_{j1}^b))^{1/j} = L < \infty \), and put \( U_0 = 0 \). Then

\[
\lim \sup_{j \to \infty} \|K0\|_{K_j}^{1/j} = \lim \sup_{j \to \infty} \left( \frac{\beta}{d} \max_{c \in A_w} \ln (1 + ac_1^b) \right)^{1/j} \\
\leq \lim \sup_{j \to \infty} \left( \frac{\beta}{d} \ln (1 + aw_{j1}^b) \right)^{1/j} \\
= L,
\]

and Theorem 4 assures that the Koopmans Equation has a unique fixed point \( U^* \) on \( A_+(w) \) for any \( \beta < \min\{1, 1/L\} \) (for instance, \( L = 1 \) when \( w = (1, \alpha, \alpha^2, \ldots) \), for \( \alpha > 1 \)). Although the utility function \( U^* \) has no explicit expression, the value of \( U^* \) on any given path can be numerically approximated by means of successive iterations of the Koopmans operator.

**Example 5 (Uzawa–Epstein–Hynes).** Let \( W(x, y) = u(x) + ye^{-v(x)} \), introduced in [19] and also studied in [8], where it is supposed that \( u \) is well defined on \( X = [0, \infty) \) and nondecreasing, \( v \) is nondecreasing, and \( v(0) > 0 \). These last two assumptions ensure, in particular, that property (W2) holds for \( \beta = e^{-v(0)} < 1 \). Now, take any increasing and unbounded sequence \( w = (w_t) \) satisfying \( \lim \sup_{j \to \infty} |u(w_j)|^{1/j} = L < \infty \), and set \( U_0 = 0 \). Then \( \lim \sup_{j \to \infty} \|K0\|_{K_j}^{1/j} = \lim \sup_{j \to \infty} |u(w_j)|^{1/j} = L \), and hence Theorem 4
assures the existence of a unique continuous fixed point $U^*$ on $A_+(w)$, that is given by

$$U^*(c) = \sum_{t=1}^{\infty} u(c_t) e^{(-\sum_{t=1}^{t-1} v(c_i))},$$

whenever $\beta < \min\{1, 1/L\}$.

**Example 6 (Assymptotic exponent of $|K_0|$ less than $\rho > 0$).** Consider an aggregator $W$ satisfying $|W(x, 0)| \leq A(1 + x_\theta)$, $A > 0$, $\rho > 0$. Take $D = \ell_+^\alpha(\alpha)$, for some $\alpha \geq 1$, and put $U_0 = 0$. Then, as it is clear that $\|K_0\|_{K_j} \leq A \max_{0 \leq c_1 \leq c_j} (1 + c_\rho) = A(1 + \alpha^j\rho)$,

it follows that

$$\limsup_{j \to \infty} \|K_0\|^{1/j}_{K_j} \leq \limsup_{j \to \infty} |A(1 + \alpha^j\rho)|^{1/j} = \alpha^\rho,$$

and, therefore, Theorem 4 applies whenever $\beta \alpha^\rho < 1$. This is the same bound that appears\(^9\) in [3].

**Example 7 (Homogeneous negative case).** Consider the homogeneous negative case in which the aggregator $W$ satisfies $|W(x, 0)| \leq b x^\theta$, for all $\forall x > 0$ for some $b > 0$, and some $\theta < 0$. Setting $D = \ell_+^\alpha(\mu, \alpha)$, for some $0 < \mu < 1 < \alpha$, we have that $|K_0(c)| \leq b(\sigma c)^\theta \leq b(\mu^j)^\theta$ for all $c \in K_j$, so that

$$\limsup_{j \to \infty} \|K_0\|^{1/j}_{K_j} \leq \limsup_{j \to \infty} |b(\mu^j)^\theta|^{1/j} = \mu^\theta.$$

Thus, Theorem 4 is applicable whenever $\beta \mu^\theta < 1$, without constraint in the value of $\alpha$. Moreover, since $\mu^\theta > 1$, the fixed point is unique on $C(\ell_+^\alpha(\mu, \alpha))$, for any $\alpha > 1$, and is also homogeneous of degree $\theta$.

**Example 8 (Logarithmic case).** Theorem 4 is also applicable to aggregators $W$ such that $a + b \ln x \leq W(x, 0) \leq A + B \ln x$ for all $x > 0$, for some nonnegative constants $a, b, A, B$.

---

\(^9\)The following facts are, however, worth noting: (i) the fixed point is continuous in the weak* topology and not only with respect to the norm $\|\cdot\|_\alpha$, and (ii) it seems that the conditions established in Theorem 4 are easier to test than those of Boyd’s Theorem, because here there is no need to find any suitable function.
Taking $D = \ell^n_+(\mu, \alpha)$, where $0 < \mu < 1 \leq \alpha$, it is clear that, for any $c \in K_j$, it follows that

$$K0(c) \geq a + b \ln \sigma c$$

$$\geq a + b \ln \mu^j$$

$$= a + bj \ln \mu.$$ 

In the same way,

$$K0(c) \leq A + B \ln \sigma c$$

$$\leq A + B \ln \alpha^j$$

$$= A + Bj \ln \alpha.$$ 

Hence, the value of $\limsup_{j \to \infty} \|K0\|_{K_j}^{1/j}$ is confined between two sequences converging to 1 and, therefore, Theorem 4 shows that, for all $\beta < 1$, a unique recursive utility function $U^*$ exists on $C(\ell^n_+(\mu, \mu))$ for any $0 < \mu < 1 \leq \alpha$. Obviously, the result extends to $A_+(v, w)$ for sequences $w, v$ satisfying $\limsup_{j \to \infty} |A + B \ln w_j|^{1/j} = L_1$, $\limsup_{j \to \infty} |a + b \ln v_j|^{1/j} = L_2$ with $\beta$ satisfying $\beta < \min\{1, 1/L_1, 1/L_2\}$.

**Example 9 (Undiscounted aggregator).** Next, an example is given in which an infinity family of fixed points exists, whenever $\beta = 1$. Consider the aggregator $W(x, y) = \delta^2 y x + \delta$, where $\delta \geq 1$. Now, for each $\alpha \in (0, 1)$, fix $D_{\alpha} = \prod_{j=1}^{\infty} [\delta^2 - \delta, \delta^2 (1 - \alpha^j) - \delta]$ as the domain. Notice that $D_{\alpha}$ is an admissible set, since it is compact, and satisfies $D_{\alpha} = \bigcup_j K_{\alpha}^j$, where $K_{\alpha}^j = \prod_{l=j}^{\infty} [\delta^2 - \delta, \delta^2 / (1 - \alpha^l) - \delta]$ (see also that $\sigma(K_{\alpha}^j) = K_{\alpha}^{j+1}$). Now observe that, on $D_{\alpha}$, the aggregator $W$ is clearly continuous and satisfies assumption (W2) for $\beta = 1$.

Taking $U_0 = \theta$ as the initial function, where $\theta$ is some real constant, it follows that

$$\lim_{j \to \infty} \|K\theta - \theta\|_{K_j^{1/j}} = \lim_{j \to \infty} \|\frac{\delta^2 \theta}{c_1 + \delta} - \theta\|_{K_j^{1/j}} = \alpha.$$ 

$\limsup_{j \to \infty} |a + bj \ln \mu|^{1/j} = \limsup_{j \to \infty} |A + Bj \ln \alpha|^{1/j} = 1.$
Then Theorem 4 applies directly assuring the existence of a fixed point $U'$, which is continuous on $D_\alpha$, whenever $\beta \alpha = \alpha < 1$. Moreover, the iterates $K^n\theta$ converge to $U'(c) = \theta \prod_{t=1}^\infty \frac{\delta^2}{c_{t+\theta}}$ in the metric $d_c$, when $n$ goes to infinity, for all $c$ satisfying $c\alpha < 1$. Now, for each $\theta \in \mathbb{R}$, consider the collection of limits of all sequences $K^n\theta$ constructed as above. An infinite number of fixed points $U_\theta'(c) = \theta \prod_{t=1}^\infty \frac{\delta^2}{c_{t+\theta}}$, all of which are continuous in $D_\alpha$ with respect to the weak* topology, are then obtained. Hence, for any $\delta \geq 1$, the corresponding aggregator $W$ has a non-countable quantity of fixed points in $D_\alpha$, for any $\alpha < 1$.

We should point out that, for $\delta < 1$, the set $D_\alpha$ is an improper domain since it does not fulfill the usual convention that consumption streams are in $\mathbb{R}_{++}^\infty$. However, it is curious to see that, whenever consumption streams are permitted in $\mathbb{R}^\infty$, the aggregator has an infinite quantity of fixed points. Consequently, the aggregator $W$ constitutes an example in which an infinite family of fixed point exists for all $\delta > 0$.

Under the usual assumption on the domain and $\delta < 1$, Theorem 4 can be used to show that the aggregator has a unique fixed point, which obviously coincides with the null function $U_0'$. To see this, it is enough to put $\theta = 0$ and to observe that

$$\limsup_{j \to \infty} \|K\theta\|^{1/j}_{K_j} = 0.$$

**Example 10 (Undiscounted aggregator II).** Another example in which different initial real seeds converge to different fixed points is the aggregator $W(x, y) = x + y$. To show this, take any constant $\theta \in \mathbb{R}$ as the initial function $U_0$, and consider, for each $\alpha < 1$, the compact domain $D_\alpha = \prod_{j=1}^\infty [0, \alpha^j]$. First note that $D_\alpha = \cup K_j^\alpha$, where $K_j^\alpha = \prod_{l=j}^\infty [0, \alpha^l]$, and notice that $\sigma(K_j^\alpha) = K_j^{\alpha+1}$. Then, observe that

$$\limsup_{j \to \infty} \|K\theta - \theta\|^{1/j}_{K_j} = \lim_{j \to \infty} \|c_1\|^{1/j}_{K_j} = \alpha.$$

Now, since $\beta\alpha = \alpha < 1$, Theorem 4 applies assuring the existence of a fixed point $U'(c) = \theta + \sum_{j=1}^\infty c_j$ on $C(D_\alpha)$, that can be calculated as the limit, with respect to the
metric $d_c$, of $K^nθ$, as $n$ goes to $∞$, for all $c$ satisfying $cα < 1$. This latter affirmation is due to the fact that distance the $d_c$ is just a contraction for values of $c$ lying in the interval $(0, 1/α)$. Hence, the collection of functions $U^*_θ = θ + U^*$ constitutes an infinity class of fixed points, as commented above.

**Example 11 (Upcounted aggregator).** Take $β > 1$, and define the upcounted aggregator $W(x, y) = x + β y v(x)$, where $v$ is any real function satisfying $v \leq v(0) = 1$. Next, choose $α > 0$ such that $βα < 1$, and set the compact domain $D_α = \prod_{j=1}^{∞}[0, α^j]$ (note again that $D_α = \bigcup K_α^j$, where $K_α^j = \prod_{l=j}^{∞}[0, α^l]$, and that $σ(K_α^j) = K_α^{j+1}$). Now put $U_0 = 0$, and observe that

$$\limsup_{j→∞} \|K_0\|_{K_j}^{1/j} = \limsup_{j→∞} \|c_1\|_{K_j}^{1/j} = \lim_{j→∞} \|α^j\|^{1/j} = α.$$ 

Then, since $βα < 1$, Theorem 4 applies on $C(D_α)$, yielding the utility function

$$U^*(c) = \sum_{t=1}^{∞} β^{t-1} c_t (\prod_{l=1}^{t-1} v(c_l)).$$

Of course, $U^*$ is weak* continuous on $D_α$, and the iterates $K^α0$ converge to $U^*$ on the aforementioned domain (the convergence is with respect to the metric $d_c$, for any $c$ satisfying $cα < 1$).

**Example 12 (Upcounted aggregator II).** The following example also illustrates the applicability of Theorem 4 for obtaining a recursive utility for upcounted aggregators. Let $W(x, y) = x^p + \beta (y - 1)$, where $p$, $β > 1$. For any $α > 0$ satisfying $βα < 1$, we can consider $D_α = \prod_{j=1}^{∞}[β^{1/j} (α^j + β)^{1/j}]$ as the domain (observe once again that $D_α = \bigcup K_α^j$, where $K_α^j = \prod_{l=j}^{∞}[β^{1/j} (α^l + β)^{1/j}]$, and that $σ(K_α^j) = K_α^{j+1}$). Now, putting $U_0 = 0$, a routine computation shows that

$$\limsup_{j→∞} \|K_0\|_{K_j}^{1/j} = \limsup_{j→∞} \|c_1^p - \beta\|_{K_j}^{1/j} = \lim_{j→∞} \|α^j\|^{1/j} = α.$$ 

\[\text{11This hypothesis assures that } W\text{ obeys the Lipschitz condition (W2), for } β(x) = β.\]
This assures that a recursive utility function $U^*$ exists (by Theorem 4), and is weak* continuous on $D_\alpha$. Since $K^n0$ converges to $U^*$ on $D_\alpha$, recursive substitution in the recursion operator yields the expression

$$U^*(c) = \sum_{t=1}^{\infty} \beta^{t-1}(c_t^{p} - \beta).$$

As a final comment it can be concluded, roughly speaking, that increases in the factor $\beta$ widely decrease the "size" of the domain in which a continuous recursive utility exists.

**Example 13 (Stochastic recursive utility).** We shall now consider the stochastic recursive utility case, which can be described by means of two components: an aggregator $W$ and a certainty equivalent $\mu: \mathbb{R}_+ \rightarrow \mathbb{R}_+$. We refer to [9] for the usual assumptions about the certainty equivalent. Following these authors, we impose the following hypotheses:

(T) $\mu$ obeys the Lipschitz condition: $|\mu(x) - \mu(y)| \leq \beta'|x - y|$, for all $x, y \in \mathbb{R}_+$.

Given both an aggregator $W$ and a certainty equivalent $\mu$, an operator $K_\mu$ can be defined as follows

$$K_\mu U(c) = W(\sigma c, \mu(U(\sigma c))). \quad (7)$$

Under assumptions (W1) and (T), $K_\mu$ maps continuous functions on continuous functions in the weak* topology. In this framework, Theorems 3 and 4 could be trivially established, by replacing $K$ with $K_\mu$ and $\beta$ with the product $\beta\beta'$.

Analogously, as in Example 6, let $W$ be an aggregator satisfying $|W(x, \mu(0))| \leq A(1 + x^\rho)$, where $A, \rho > 0$. In [13] it is imposed further assumptions of differentiability, monotonicity and concavity on the aggregator, and then is shown that a unique fixed point for the operator $K_\mu$ exists on $\ell_+^N(\alpha)$, whenever $\beta\beta'\alpha^\rho < 1$. As we shall see now, the additional properties considered in [13] are not necessary to show this fact. To do so, it is sufficient to consider $D = \ell_+^N(\alpha)$, with $\alpha > 1$, and to choose $U_0 = 0$. Then

$$\limsup_{j \to \infty} \frac{\|K_\mu 0\|_{K_j}^{1/j}}{K_j} \leq \limsup_{j \to \infty} \frac{\|A(1 + c_t^{p})\|_{K_j}^{1/j}}{K_j} = \limsup_{j \to \infty} \frac{\|A(1 + \alpha^{\rho j})\|_{K_j}^{1/j}}{K_j} = \alpha^\rho,$$
and, from Theorem 4, a unique fixed point $U'$ exists on $C(\ell^2_+)$, whenever $\beta\beta'\alpha^\rho < 1.$
APPENDIX

PROOFS

Proof of Theorem 3:

(a) We first prove that the Koopmans operator $K: D \to D$ is a 0–LC. Let $U, V \in C(D)$ and let $c \in K_j$. Then

\[
|KU(c) - KV(c)| = |W(\pi c, U(\sigma c)) - W(\pi c, V(\sigma c))| \\
\leq \beta(\pi c) |U(\sigma c) - V(\sigma c)| \quad \text{(from (W2))} \\
\leq \max_{c \in K_j} \beta(\pi c) |U(\sigma c) - V(\sigma c)| \\
\leq \beta_j \max_{c \in K_j} |U(c) - V(c)| \quad \text{(since } \sigma K_j \subseteq K_j) \\
= \beta_j \|U - V\|_{K_j}.
\]

Thus, we have $\|KU - KV\|_{K_j} \leq \beta_j \|U - V\|_{K_j}$ and $K$ is a 0–LC. On the other hand, define the set $A \subseteq D$ as follows

\[
A = \left\{ U \in C(D) : \|U\|_{K_j} \leq \frac{\|K0\|_{K_j}}{1 - \beta_j} \text{ for all } j \in \mathbb{N} \right\}.
\]

Then $A$ is a closed and bounded subset of $C(D)$, and for all $U \in A$ it follows that

\[
\|KU\|_{K_j} \leq \|K0\|_{K_j} + \beta_j \|U\|_{K_j} \leq \|K0\|_{K_j} + \beta_j \frac{\|K0\|_{K_j}}{1 - \beta_j} = \frac{\|K0\|_{K_j}}{1 - \beta_j} \text{ for all } j \in \mathbb{N}.
\]

This means that the Koopmans operator maps $A$ into $A$, and consequently, from Theorem 1, it is a contraction on $A$. Hence, the Koopmans operator has a unique fixed point $U^*$ on $A$. The uniqueness of the fixed point on $C(D)$ again follows from Theorem 1.

(b) It follows from (ii) of Theorem 1.

\[
Q.E.D.
\]
Proof of Theorem 4:

(a) We first prove that the Koopmans operator $\mathcal{K} : C(D) \rightarrow C(D)$ is a 1–LC. Let $U, V \in C(D)$ and let $c \in K_j$. Then

\[
|\mathcal{K}U(c) - \mathcal{K}V(c)| = |W(Ic, U(\sigma c)) - W(Ic, V(\sigma c))| \\
\leq \beta |U(\sigma c) - V(\sigma c)| \quad \text{(from (W2))} \\
\leq \max_{c \in K_j} \beta |U(\sigma c) - V(\sigma c)| \\
\leq \beta \|U - V\|_{K_{j+1}} \quad \text{(since $\sigma K_j \subseteq K_{j+1}$)}.
\]

Thus, we have $\|\mathcal{K}U - \mathcal{K}V\|_{K_j} \leq \beta \|U - V\|_{K_{j+1}}$ and $\mathcal{K}$ is a 1–LC.

Now let $U, V \in C(D)$. Then, since $\mathcal{K}$ is a 1–LC, we obtain

\[
d_c(\mathcal{K}U, \mathcal{K}V) = \sum_{j=1}^{\infty} c_j \|\mathcal{K}U - \mathcal{K}V\|_{K_{j}} \\
\leq \sum_{j=1}^{\infty} c_j \beta \|U - V\|_{K_{j+1}} \leq \frac{\beta}{c} d_c(U, V). \quad (8)
\]

From this, we know that metric $d_c$ is a contraction for those continuous functions $U, V$ satisfying $d_c(U, V) < \infty$.

On the other hand, define the set $\mathbb{A} \subseteq C(D)$ as follows

\[
\mathbb{A} = \left\{ U \in C(D) : d_c(U, U_0) \leq \frac{d_c(\mathcal{K}U_0, U_0)}{1 - \beta/c} \right\}.
\]

Then, $\mathbb{A}$ is closed and bounded. Next, we prove the Koopmans operator maps $\mathbb{A}$ into itself. But this is immediate, since, for $U \in \mathbb{A}$, we have

\[
d_c(\mathcal{K}U, U_0) \leq d_c(\mathcal{K}U, \mathcal{K}U_0) + d_c(\mathcal{K}U_0, U_0) \\
\leq \frac{\beta}{c} d_c(U, U_0) + d_c(\mathcal{K}U_0, U_0) \quad \text{(from (8))} \\
\leq \frac{\beta}{c} \frac{d_c(\mathcal{K}U_0, U_0)}{1 - \beta/c} + d_c(\mathcal{K}U_0, U_0) \\
= \frac{d_c(\mathcal{K}U_0, U_0)}{1 - \beta/c},
\]

which means that $\mathcal{K}U$ belongs to $\mathbb{A}$. 29
Hence, since $\beta < c$, we conclude that the Koopmans operator is a contraction on $A$ with respect to the metric $d_c$ and, therefore, has a unique fixed point $U^*$ on $A$.

(b) Since $K$ is a contraction on $A$, $K^n U \xrightarrow{d_c} U^*$ as $n \to \infty$ for all $U \in A$. $Q.E.D.$
References


