Long-lived Assets and Optimality*

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ABSTRACT

We consider general OLG economies under uncertainty, with dividend paying assets of infinite maturity and money, and in which one good is available for consumption. We study the optimality properties of equilibria with asset markets. Our notion of optimality takes into account the restrictions faced by agents in making state contingent transfers. We provide a complete characterization, in terms of the prices and dividends of assets of infinite maturity, of those equilibria that are optimal; we show that for suboptimality to obtain, one needs a condition in addition to a generalized Cass criterion. Results for various special cases follow; in particular, we show that if dividends are non-negligible in a weak sense and assets are freely disposable, then every non-monetary equilibrium allocation is optimal. Other results shed light on the special role played by money vis-a-vis other assets of infinite maturity in determining the optimality properties of equilibria.

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1. INTRODUCTION

We consider a general class of pure exchange overlapping generations (OLG) economies under uncertainty in which one good is available for consumption in each period and intertemporal transfers are carried out by trading in assets. Our objective is to analyze the role of long-lived assets (dividend paying and money), which can have negative payoffs and need not be freely disposable, in ensuring optimality of competitive allocations.

The motivation for our exercise comes from many sources. The fact that competitive OLG economies need not allocate resources efficiently is well known and is aptly summarized in the “chocolate” parable where more can be consumed today by making every young agent give one chocolate to a contemporaneously old agent. This inefficiency occurs even in the most basic deterministic model and has been characterized.\(^1\) One uses a Pareto criterion and one assumes that the only restriction that the planner faces is that on aggregate resources so that arbitrary lump sum transfers between generations can be used. The Pareto criterion used under uncertainty is called *conditional Pareto optimality* (CPO), proposed by Muench (1977), and is one in which agents are distinguished not only according to their type and their date of birth but also according to the event at that date. In all the cases, the result is in the form of a Cass criterion where contingent commodity prices (or discounted prices) appear.\(^2\)

Abel, Mankiw, Summers and Zeckhauser (1989)—henceforth, AMSZ—tackled a closely related problem. They consider an OLG economy under uncertainty with a single agent and a neoclassical technology and provide an easy to use condition, the net dividend criterion, which is sufficient to render an equilibrium allocation CPO.\(^3\) Somewhat surprisingly, one can construct examples of deterministic economies with a dividend paying asset which satisfy the net dividend criterion but for which the Cass sum converges. This would seem to contradict the results for the deterministic OLG model!\(^4\)

Wilson (1981) studied an economy with contingent commodities and showed that the presence of a nonnegligible individual forces the value of the aggregate endowment to be finite and the equilibrium to be Pareto optimal. The same holds if there is a long-lived

\(^1\)In OLG economies under certainty by Okuno and Zilcha (1980) and Balasko and Shell (1980), Geanakoplos and Polemarchakis (1991) for the case in which markets are complete, and Chattopadhyay and Gottardi (1999) for the case in which markets are sequentially complete, i.e., when agents are able to insure against all sources of uncertainty affecting them after their birth.

\(^2\)Theorems 1 and 2 in Demange (2002) provide sufficient conditions for CPO with sequentially complete markets and many periods of life. In the case of two period lifetimes, the results follow from Theorem 3 in Chattopadhyay and Gottardi (1999).

\(^3\)They use the term dynamic efficiency instead of CPO. We prefer to use the term CPO since dynamic efficiency is often used to refer just to the production side of the economy.

\(^4\)AMSZ do not work with the Cass criterion; instead they solve a planner’s problem in which the planner chooses state dependent quantities of capital. They verify that a transversality condition proposed by them is satisfied when the net dividend criterion holds. This, together with the fact that the other conditions for intertemporal optimality are satisfied by an equilibrium allocation, leads them to their conclusion.
dividend paying asset with a sufficiently large dividend.\textsuperscript{5}

The AMSZ result suggests that the formulation of the problem in terms of a sequence of markets rather than as a contingent commodity market might have consequences. Furthermore, criteria in the form of dividends and asset prices are, at least in principle, observable while contingent commodity prices need to be constructed. Unfortunately, the AMSZ result is very partial; also, it does not address a key case, that in which the asset in question is money, since both the results that they state are in terms of the dividend as a fraction of the value of the asset where the fraction is positive for sufficiency and negative for necessity.

Our interest in looking at OLG models with long-lived assets stems from these earlier contributions. At a more applied level, the model that we consider is the canonical framework for the analysis of intertemporal risk sharing and the results obtained provide insights in a wide range of applications; an evident one is the problem of identifying desirable features of any reform of social security systems.

In our model agents live for two periods and there is only one consumption good. Assets are unrestricted in terms of the sign of the dividends and they need not be freely disposable.\textsuperscript{6} We think of assets as reduced forms of the right to use a technology, where buying an asset today requires a commitment to investing a certain amount in every event tomorrow; the gross return is random and could be low enough to induce a negative net dividend tomorrow and this justifies the consideration of a general dividend process (Brock’s example reminds us that free disposability of assets can be an essential restriction).\textsuperscript{7} Although the case where markets are sequentially complete is our leading case, we also consider what happens with sequential incompleteness. Hence, the set of assets available is allowed to be sparse so that agents might not be able to overcome

\textsuperscript{5}Scheinkman (1980) made a similar point. Santos and Woodford (1997) extended the result to a multi-good model with sequentially complete asset markets when dividends are non-negative and assets are freely disposable.

Allen and Gale (1997) studied a model with one good and one agent in every period, so that any equilibrium allocation obtained with a sequentially complete market can be induced with just one asset, and showed that the existence of a dividend paying asset implies that the resulting stationary equilibrium allocation is optimal.

Theorem 3 in Demange (2002) also provides a result which applies when the dividend is uniformly positive and the asset is freely disposable.

\textsuperscript{6}Santos and Woodford (1997) noted the importance of imposing free disposal of assets and non-negativity of dividends in order to derive their results on the non-existence of bubbles; the restriction is essential since an example due to Brock (1990) showed how a standard economy can have a stationary equilibrium in which the price of the asset is negative.

\textsuperscript{7}This also agrees with the approach in AMSZ who think of dividends quite generally. However, we do not have investment in physical capital; instead we have “trees”. This is in part because the framework is less messy, in part because with incomplete markets it is not clear that the firm will want to maximize profits, and in part because all of our results are in terms of marginal conditions which should not depend too much on the underlying framework. Furthermore, at least in the case of sequentially complete markets and contingent commodities, Barbie, Kaul, and Hagedorn (2002) show that introducing capital accumulation with a bounded neoclassical technology does not alter in any way the characterization result for the pure exchange case obtained by Chattopadhyay and Gottardi (1999).
the fact that they face a sequence of budget constraints. In order to reduce notation we assume that the same set of assets is available at different nodes in a date-event tree.

The model covers three broad categories of long-lived assets: the case where the long-lived asset can be freely disposed and pays a non-negative dividend which is sometimes positive, land being the standard example; the case where the long-lived asset never pays a dividend, i.e., fiat money; and cases where free disposal fails and/or the dividend is not required to be non-negative. Also, we assume that the endowment of each of the assets is independent of time and the realization of uncertainty, i.e., a constant. For dividend paying assets it is reasonable to assume that the endowment never changes (one can redo the analysis allowing for endowment growth). When dealing with money this appears to be unreasonable; however, since our results are in terms of asset prices and creating money cannot change the physical resource constraint, we think that allowing for active monetary policy will not affect the results.

Our welfare criterion is CPO but we restrict the transfers that the planner can use in following way: we say that a lump sum tax/subsidy scheme is feasible if there is a way to specify portfolios for every agent such that, given dividends and asset prices, the return to holding the portfolio in every state when old coincides with the tax/subsidy that the agent was supposed to pay/receive. This captures the idea that the planner must face the same constraints as the agents and must use the returns to existing assets to make transfers though there is no actual intervention in asset markets. The returns are specified by asset prices which are assumed to remain at their initial equilibrium values.

The assumption that asset prices do not change is not controversial when markets are sequentially complete. In that case our welfare criterion singles out the special role played by dividend paying assets in allowing for net intergenerational transfers over and above their role in completing the market. However, when markets fail to be sequentially complete, agents will anticipate the transfers that they receive and will react to them by changing their asset demands so that assuming that asset prices do not change is no longer innocuous since market incompleteness could have bite.\textsuperscript{8} Evidently, it is desirable to allow price effects; unfortunately, the problem is extremely difficult. There are a few partial results which identify situations in which a form of constrained optimality prevails inspite of the price effect.\textsuperscript{9} In the case of sequentially incomplete markets, we think of

\textsuperscript{8}Geanakoplos and Polemarchakis (1986) showed that in two-period general equilibrium models with incomplete markets, equilibrium allocations are generically constrained suboptimal provided that two or more goods are traded. The notion of optimality used is that of Pareto comparisons between allocations that can be induced as equilibria with trade in spot markets but with portfolios that are assigned by the planner.

\textsuperscript{9}Cass, Green, and Spear (1992) showed that there are no locally improving stationary asset redistributions which improve over the stationary monetary equilibria of a one-good stochastic OLG economy with incomplete asset markets, no dividend paying long-lived asset, and freely disposable money, when the price of money is allowed to adjust. Demange (2002) showed that constrained optimality obtains when asset prices are allowed to adjust without the restriction to local price changes but assuming free disposal of assets and dividends which are uniformly positive.
our work as partial and a first approach to analysing how well dividend paying assets perform in terms of intertemporal optimality; including price effects can only indicate further inefficiencies.

Our main results, Theorems 1 and 2, provide necessary and sufficient conditions for a competitive allocation to be constrained optimal. The results say that a competitive allocation is not optimal if and only if there is a set of histories, and an asset portfolio, such that, on every history in the set, the discounted value of the portfolio is positive and uniformly bounded and the discount factors are also uniformly bounded. The discounted value plays the role of a Cass type criterion and the discount factors used are event dependent (random discounting) and defined in terms of the returns at each event to the portfolio.

The AMSZ net dividend criterion implies that the discount factors we define do not satisfy the appropriate bound even though the Cass sum might converge. Therefore, their condition is sufficient for CPO even though the Cass sum might indicate otherwise. This allows us to clarify the nature of our contribution which combines the insights from earlier analyses in terms of contingent commodity markets with the AMSZ approach of thinking in terms of asset prices to obtain a pair of conditions violation of any one of which leads to optimality.

We provide results for various special cases with only one long-lived asset including the case in which the equilibrium is stationary; these results follow easily from the characterization and shed light on the range of situations covered by the result. An important implication of the main result is that, in a non-monetary economy, an equilibrium allocation with free disposal of assets and non-negative dividends which are positive infinitely often on every history (which is a much weaker version of nonnegligibility) is necessarily constrained optimal. Another implication is that if money is the only long-lived asset then all stationary monetary equilibria are constrained optimal. Our results also shed light on the Chattopadhyay and Gottardi (1999) characterization.

There is a problem within the framework that we develop that we have been unable to solve. Consider an economy with a dividend paying asset and money where it is known that a constrained improvement in our sense cannot be constructed with the dividend paying asset alone. Is it possible that money prices are so badly behaved that an improvement can be constructed using only money? Our characterization result does apply to the framework with many assets but is silent on this issue.

More generally one would like to do away with the assumption regarding fixed asset prices but that is a very challenging question.

The rest of the paper is structured as follows. Section 2 presents the model and notation. In Section 3 we present a definition of constrained optimality. In Section 4 we state and discuss our main result. Various special cases of interest and relation to earlier work are discussed in Section 5 which can be read after glancing through Section 4. The proofs of the two theorems are relegated to Section 6.
2. THE MODEL

We consider a general pure exchange overlapping generations (OLG) economy under uncertainty where only one consumption good is traded and agents live for two periods. The economy evolves in discrete time with uncertainty in the environment described by an abstract date-event tree as in Chapter 7 of Debreu (1959). We turn to a formal description of the model and the notation used.

Time is discrete and dates are denoted $t = 1, 2, 3, \cdots$.

Let $\mathcal{S} = \{1, 2, \cdots, S\}$ be a state space, the set from which a state is chosen at each date; so $\#\mathcal{S} = S$. The structure of the date-event tree induced by all possible realizations of states from an initial date $t = 0$ is as follows. The root of the tree is $\sigma_0 \in \mathcal{S}$; $\Sigma_t$ is the set of nodes at date $t$ where we set $\Sigma_1 := \{\sigma_0\} \times \mathcal{S}$, and iteratively set $\Sigma_t := \Sigma_{t-1} \times \mathcal{S}$ for $t = 2, 3, \cdots$. Define $\Sigma := \bigcup_{t \geq 1} \Sigma_t$ and $\Gamma := \{\sigma_0\} \cup \Sigma$. Elements of $\Gamma$ are called nodes (to be thought of as the “date-events” or simply “events”), and a generic node is denoted $\sigma$. Given a node $\sigma \in \Sigma$, $t(\sigma)$ denotes the value of $t$ at which $\sigma \in \Sigma_t$. Clearly, a node $\sigma \in \Sigma_t$ is nothing but a string of states $(\sigma_0, s_1, s_2, \cdots, s_t)$ where $s_\tau \in \mathcal{S}$ denotes the state realized at date $\tau$, $\tau = 1, \cdots, t$ ($\sigma_0$ is the realization at the initial date). It follows that the predecessor of a node $\sigma \in \Sigma_t$ is uniquely defined and it will be denoted by $\sigma_{-1}$, an element of $\Sigma_{t-1}$; the set of immediate successor nodes of a node $\sigma$ is denoted $\sigma^+$. A path is defined by an infinite sequence of nodes $\{\sigma_t\}_{t \geq 1}$ such that, for all $t \geq 1$, $\sigma_t$ is the predecessor of $\sigma_{t+1}$; $\sigma^\infty$ will denote a path.

One commodity is available for consumption at each node $\sigma \in \Sigma$.

At each node $\sigma \in \Sigma$, a generation of agents, denoted $\mathcal{H}$, is born, where $H := \#\mathcal{H}$. A member of generation $\sigma$ of type $h \in \mathcal{H}$ is denoted $(\sigma, h)$. In addition, there is a set, $\mathcal{I}$, of $H$ agents who enter the economy at each node $\sigma \in \Sigma_1$ at date 1; they constitute the generation of the “initial old”, and are denoted $(\sigma; o, h)$, where $\sigma \in \Sigma_1$. The set of agents is denoted $\mathcal{I}$ where $\mathcal{I} := (\Sigma_1 \times \{o\} \times \mathcal{H}) \cup (\Sigma \times \mathcal{H})$. $\sigma(i)$ identifies the node at which $i$ was born; so $\sigma(i) \in \Sigma$ with $\sigma(i) \in \Sigma_1$ for the initial old.

We assume that the initial old live only at the node at which they enter the economy, while every other agent is alive at the node of birth and at every node which is an immediate successor of the node at which she was born, $\{\sigma(i), \sigma^+\}$. The set of agents alive at a node $\sigma$ is denoted $\mathcal{I}(\sigma)$ where $\mathcal{I}(\sigma) := \{i \in \mathcal{I} : \sigma \in \Sigma_i\}$.

Each agent $i \in \mathcal{I}$ is described by a consumption set, $X_i \subset R$ or $X_i \subset R^{1+S}$, an endowment vector, $\omega_i = (\omega_i(\sigma))_{\sigma \in \Sigma_i} \in X_i$, and a utility function, $u_i : X_i \to R$.

There is a set $\mathcal{J} = \{1, 2, \cdots, J\}$ of one-period lived short maturity assets, with payoffs (per unit) in the commodity described by the function $s : \Sigma \to R^J$. There is also a set of dividend paying assets of infinite maturity, denoted $\mathcal{K} = \{1, 2, \cdots, K\}$, with payoffs (per unit) in the commodity specified by the function $d : \Sigma \to R^K$. Finally, we consider an asset of infinite maturity called fiat money, denoted $m$. Fiat money is characterized by the fact that it never pays a dividend. The set of assets available is $\mathcal{A} := \mathcal{J} \cup \mathcal{K} \cup \{m\}$.

Only the initial old are endowed with these assets and their endowments are denoted $\omega^0(i)$, $a \in \mathcal{A}, i \in \Sigma_1 \times \{o\} \times \mathcal{H}$. A negative endowment of an asset indicates a pre-existing
We assume that, for every asset, the total endowment is independent of the node at date \( t = 1 \). Define \( \omega^a := \sum_{i \in \Sigma_1 \times \{0\} \times \mathcal{H}} \omega^a(i) \) for \( a \in \mathcal{A} \), the total endowment of each asset.

We introduce a notational convention. For \( z \in R^{J+K+1} \) we write \( z = (z_s, z_d, z_m) \) where \( z_s := (z^1, z^2, \ldots, z^J) \), \( z_d := (z_1^2, \ldots, z^K) \), and \( z_m := z^m \). So \( \omega = (\omega_s, \omega_d, \omega_m) \), \( \omega \in R^{J+K+1} \), gives the total endowment of each asset.

An asset is an inside asset if its total endowment is zero, i.e., it is in zero net supply.

Denoting \( \omega(\sigma) \) the total endowment of commodities at node \( \sigma \), we have:
\[
\omega(\sigma) := \sum_{i \in \mathcal{I}(\sigma)} \omega_i(\sigma) + \omega_s \cdot s(\sigma) + \omega_d \cdot d(\sigma) \quad \text{for } \sigma \in \Sigma.
\]

We assume

**ASSUMPTION 1:**
(i) \( 1 \leq H < \infty \), \( 1 \leq S < \infty \), \( 0 \leq \# \mathcal{A} < \infty \).
(ii) For all \( i \in \Sigma_1 \times \{0\} \times \mathcal{H} \), \( \omega^a(i) \in R \) for all \( a \in \mathcal{A} \), with \( \omega_s = 0 \) and with \( \omega_d \geq 0 \).\(^{10}\)
(iii) For all \( i \in \Sigma_1 \times \{0\} \times \mathcal{H} \), \( X_i = R_+ \), for all \( i \in \Sigma \times \mathcal{H} \), \( X_i = R_+^{1+S} \),
\[
\omega(i) := \sum_{i \in \mathcal{I}(\sigma)} \omega_i(\sigma) + \omega_s \cdot s(\sigma) + \omega_d \cdot d(\sigma)
\]
\[
\text{for } \sigma \in \Sigma.
\]

We have imposed monotonicity and a differentiable form of strict quasi-concavity of utility functions, and the condition that every commodity is available in a strictly positive quantity. We have also imposed the condition that the short maturity assets are inside assets, a natural restriction, and that the dividend paying assets are available in non-negative quantities; we have not imposed any restriction on the sign of the total amount of money in the economy or on the sign of the dividend.

A consumption plan for agent \( i \) is denoted \( x_i = ((x_i(\sigma))_{\sigma \in \Sigma_1} ) \in X_i \).

The next definition is standard and specifies the set of feasible allocations.

**DEFINITION 1:** A feasible allocation \( x \) is given by an array \( ((x_i)_{i \in \mathcal{I}}) \) such that
\[
x_i \in X_i \text{ for all } i \in \mathcal{I} \quad \text{and} \quad \sum_{i \in \mathcal{I}(\sigma)} x_i(\sigma) \leq \omega(\sigma) \text{ for all } \sigma \in \Sigma.
\]

We now introduce the notion of equilibrium. Given the nature of the problem, it is easy to see that the price of the commodity can be normalized to 1 at every node. Asset prices are denoted \( q = (q_s, q_d, q_m) \in R^{J+K+1} \). The vector valued function specifying all asset prices is denoted \( q : \Sigma \rightarrow R^{J+K+1} \).

Given \( q \), the vector of asset returns \( r(\sigma) := (s(\sigma), q_d(\sigma) + d(\sigma), q_m(\sigma)) \) can be specified. This induces \( r : \Sigma \rightarrow R^{J+K+1} \), the vector valued function specifying all asset returns. Evidently, \( r \) is a function of \( q \), a fact which we suppress in the notation so as not to clutter it.

\(^{10}\)The notation \( 0 \) denotes the vector \((0, 0, \cdots, 0)\) in a space of conformal dimension.
Agents choose their asset holdings and consumption levels at a node using their endowments at that node and returns on assets carried over from the previous node. For \( i \in I \), let \( \theta(i) \in R^{I+K+1} \), denote an agent’s portfolio. Let \( \theta = ((\theta(i))_{i \in I}) \) denote the array which specifies asset holdings for all agents.

We can now define a competitive equilibrium. It requires that the allocation of commodities is feasible, that the allocation of assets clears asset markets exactly, and that agents optimize subject to a multiplicity of budget constraints.

**DEFINITION 2 (CE-S):** \((x^*, \theta^*, q^*, r^*)\) is a competitive equilibrium with a sequence of markets (CE-S) if:

(i) \( x^* \) is a feasible allocation;

(ii) for all \( \sigma \in \Sigma \), \( \sum_{h \in H} \theta^*(\sigma, h) = \omega \);

(iii) for all \( \sigma \in \Sigma \), \( r^*(\sigma) := (s(\sigma), q^*_d(\sigma) + d(\sigma), q^*_m(\sigma)) \);

(iv) for all \( i \in \Sigma_i \times \{0\} \times \mathcal{H}_i \), \( \theta^*(i) = \omega(i) \) and \( x^*_i = \omega_i + \theta^*(i) \cdot r(\sigma(i)) \);

(v) for all \( i \in \Sigma \times \mathcal{H}_i \),

(a) \( x^*_i(\sigma(i)) + \theta^*(i) \cdot q^*(\sigma(i)) \leq \omega_i(\sigma(i)) \), \( x^*_i(\sigma) \leq \omega_i(\sigma) + \theta^*(i) \cdot r^*(\sigma) \) for all \( \sigma \in \sigma(i)^+ \);

(b) if \( u_i(x) > u_i(x^*_i) \) for \( x \in X_i \), then \( x(\sigma(i)) + \theta^*(i) \cdot q^*(\sigma(i)) > \omega_i(\sigma(i)) \) or \( x(\sigma) > \omega_i(\sigma) + \theta^*(i) \cdot r^*(\sigma) \) for some \( \sigma \in \sigma(i)^+ \).

**REMARK 1:** We have imposed the condition that all asset markets must clear exactly, i.e., we have not allowed for free disposal of assets. We will also treat equilibria with free disposal of assets in which the obvious changes are made to (ii) in Definition 2 (market clearing with a weak inequality, non-negativity of asset prices, and complementary slackness) and non-negativity of the dividends is imposed as an additional condition. Also, the definition of equilibrium applies even when markets are sequentially complete, that is, if at every node the returns from the \( J + K + 1 \) assets span \( R^S \).

**REMARK 2:** The model developed above appears to be special to the extent that (i) the number of agents born at each node, (ii) the number of nodes that succeed any given node, and (iii) the number of assets of each type and their total endowment, are all taken to be independent of the node. This is without loss of generality as all our results go through with a more general specification but at the cost of more notation. Furthermore, the notation and definitions extend in a straightforward manner to the case in which \( L \) consumption goods are traded at each node and asset payoffs are denominated in the first good, and the case with a more general demographic structure.

**REMARK 3:** The optimization problem solved by an agent \( i \) can be written as

\[
\max_{x, \theta} u_i(x, (\omega_i(\sigma) + \theta \cdot r^*(\sigma))_{\sigma \in \sigma(i)^+})
\]

subject to: \( x + \theta \cdot q^*(\sigma) \leq \omega_i(\sigma(i)) \).

So each agent, effectively, solves an optimization problem with a single budget constraint and will meet the constraint with equality. This property leads to the constrained opti-
mality of all equilibria in two-period economies in which one consumption good is traded in each state.

We close the section by introducing a definition of optimality which is used in the case where markets are sequentially complete. Applying the notion of Pareto efficiency to the economy described above, where agents are distinguished by the event at their birth, yields the criterion of conditional Pareto optimality, first proposed by Muench (1977):

**DEFINITION 3 (CPO):** Let \( x \) be a feasible allocation. \( x \) is *conditionally Pareto optimal (CPO)* if there does not exist another feasible allocation \( \hat{x} \) such that

(i) for all \( i \in I \), \( u_i(\hat{x}_i) \geq u_i(x_i) \),

(ii) for some \( i' \in I \), \( u_{i'}(\hat{x}_{i'}) > u_{i'}(x_{i'}) \).

A CPO allocation requires that the risk borne in the second period of the agents’ lives be allocated optimally.\(^{11}\)

This completes the description of the model. Stationary equilibria of stationary economies constitute a special case which will be briefly developed in Section 5.

### 3. A definition of constrained optimality

Our objective is to gauge the intertemporal efficiency properties of competitive equilibria when transfers of income across dates and events are carried out via trades in asset markets. We think of inefficiency as a situation in which there are lump sum taxes that permit Pareto improving transfers of resources from one generation to another. Evidently, if the planner is unrestricted in terms of the instruments available, he could complete the market even when asset markets are not sequentially complete. It follows that we need to constrain the lump sum taxes available to the planner so that they are compatible with the restrictions that agents face in the form of the assets that exist in the economy. Once we have specified the constraints on the set of lump sum taxes that are available, we test for optimality by carrying out CPO comparisons between the equilibrium allocation and alternative allocations which differ only by the lump sum taxes.

Our definition of constrained optimality consists of the following. The planner specifies a set of lump sum taxes which are net transfers between agents alive at a certain date-event and hence they are resource feasible. These transfers must be asset market feasible in the following sense: for each young agent there is a portfolio of assets, called a variation, such that if the agent were to be allocated the variation, the state contingent payoff induced by the returns to holding the variation is exactly equal to the lump sum transfers that the planner has chosen for that agent. We will impose the restriction that the variation be “small” in keeping with the idea that all changes are local. Importantly, there is no

\(^{11}\)For a complete characterization of those competitive equilibrium allocations that are CPO when markets are sequentially complete and trade takes place in contingent commodities with many goods at each date, see Chattopadhyay and Gottardi (1999, Theorems 1 and 2).
aggregate feasibility requirement on the variations since there is no actual intervention in asset markets; nonetheless, transfers are restricted by the market. The returns to holding the various assets are determined by the dividends and asset prices. We assume that asset prices remain unchanged at their equilibrium values. This last feature is not problematic in our leading case where markets are sequentially complete.

The only assets that will be relevant are the infinite-lived ones since net intergenerational transfers cannot be achieved with assets in zero net supply.

We believe that the notion of constrained efficiency that we have defined aptly captures the problem of intertemporal improvements under uncertainty when asset markets may or may not be complete. An alternative specification would be in terms of an actual portfolio reallocation but in that case, when markets are sequentially incomplete, the change in equilibrium asset prices must be taken into account. A detailed analysis of that case is required but, as we noted in the Introduction, such an analysis in a general framework has proved elusive so far and must be left for future research.

DEFINITION 4: Given $x$, a feasible allocation, and $q : \Sigma \to R^{J+K+1}$, a specification of asset prices, $\hat{x}$ is $q$-constrained feasible, if there exists a variation $((\Delta \theta(i))_{i \in I})$, with $\Delta \theta_s(i) = 0$ for all $i \in I$, such that:

(i) for all $\sigma \in \Sigma_1$, for all $h \in H$, $-1 \leq \Delta \theta^a(\sigma;o,h) \leq 1$ for all $a \in K \cup \{m\}$;

(ii) for all $\sigma \in \Sigma$, for all $h \in H$, $-1 \leq \Delta \theta^a(\sigma,h) \leq 1$ for all $a \in K \cup \{m\}$;

(iii) for all $i \in \Sigma_1 \times \{o\} \times H$, $\hat{x}_i = x_i + \Delta \theta(i) \cdot r(\sigma)$;

(iva) for all $\sigma \in \Sigma$, $\sum_{h \in H} \hat{x}_{\sigma,h}(\sigma) = \sum_{h \in H} x_{\sigma,h}(\sigma) - \sum_{h \in H} \Delta \theta(\sigma^-1) \cdot r(\sigma)$;

(ivb) for all $i \in \Sigma \times H$, $\hat{x}_i(\sigma) = x_i(\sigma) + \Delta \theta(i) \cdot r(\sigma)$ for all $\sigma \in \sigma(i)^+$.

There is only one element in Definition 4 that needs further comment. In (iva) we only require the aggregate feasibility of consumption by young agents without pinning down consumption levels for each agent. As we shall see, this suffices for our purposes.

DEFINITION 5: An allocation $x$ is $q$-constrained CPO if there is no $\hat{x}$ which is $q$-constrained feasible and a CPO improvement over $x$.

This completes the discussion of the notion of constrained optimality that we shall use; its implications are the subject of the next two sections.

4. A COMPLETE CHARACTERIZATION RESULT

In this section we state our main result; we provide a complete characterization, in terms of asset prices and dividends, of those competitive equilibrium allocations that are optimal under a criterion that takes into account the fact that trades are carried out using assets, that markets may be incomplete, and that dividends and asset prices may be negative (Definition 5). The result also applies when markets are sequentially complete and/or assets are freely disposable.
Our result is a criterion of the type first obtained by Cass (1972) and it is well known that a pair of curvature conditions are an essential ingredient in generating the result; for sufficiency one needs to impose a uniformity condition that the curvature of every agent’s upper contour set at the competitive allocation exceeds a strictly positive number (the condition is slightly stronger than differentiable strict quasi-concavity of the utility function of every agent) while for necessity one needs to impose a finite upper bound on the curvature of every agent’s upper contour set at the competitive allocation. We use \( \underline{\rho}_{i} \) and \( \bar{\rho}_{i} \) to denote the greatest lower bound and the least upper bound, respectively, on the curvature of an agent’s upper contour set at the competitive allocation.\(^{12}\)

We introduce a bit of notation. Given an equilibrium \((x^*, \theta^*, q^*, r^*)\) and a function \(f : (\Sigma_1 \times \{o\}) \cup \Sigma \to \mathbb{R}^{J+K+1}\), induce the functions \(P_f : \Sigma \to \mathbb{R}\) and \(C_f : \Sigma \to \mathbb{R}\)

\[
P_f(\sigma) := f(\sigma, o) \cdot r^*(\sigma) \quad \text{for all } \sigma \in \Sigma_1 \\
C_f(\sigma) := f(\sigma) \cdot q^*(\sigma) \quad \text{for all } \sigma \in \Sigma.
\]

The average of variations of agents born at the same node specifies the function \(f\) where the initial old are explicitly taken into account; \(P_f\) identifies the payoff from the average variation at different nodes, while \(C_f\) identifies the cost of the average variation. Of course, there is no way to set the cost of the variation to the initial old.

We need two additional concepts. Given \(\tilde{\sigma} \in \Gamma\), we define a sub-tree (of the tree \(\Gamma\)) with root \(\tilde{\sigma}\), denoted \(\Gamma_{\tilde{\sigma}}\), as a collection of nodes such that \(\Gamma_{\tilde{\sigma}}\) itself is a tree with \(\tilde{\sigma}\) as its root. A path in the sub-tree \(\Gamma_{\tilde{\sigma}}\), denoted \(\sigma^\infty(\Gamma_{\tilde{\sigma}})\), is a collection of nodes which are ordered by precedence and for \(t \geq t(\tilde{\sigma})\) all the nodes are elements of the sub-tree, i.e., \(\sigma^\infty(\Gamma_{\tilde{\sigma}}) \subset \{\sigma^\infty_1, \sigma^\infty_2, \ldots, \sigma^\infty_{t(\tilde{\sigma})-1}\} \cup \Gamma_{\tilde{\sigma}}\), where \(\sigma^\infty_t\) denotes the \(t\)th coordinate of the path.

We can now state our characterization result.\(^{13}\)

**THEOREM 1 (Sufficiency):** Let \((x^*, \theta^*, q^*, r^*)\) be a competitive equilibrium with a sequence of markets (CE-S) and suppose Assumption 1 holds. Assume that \(x^*_i \in \mathbb{R}^{1+J_S}\), for all \(i \in \Sigma \times \mathcal{H}\), and that there are real numbers \(\Omega > 0\) and \(\bar{\rho} > 0\) such that

(i) \(\omega(\sigma) \leq \Omega\) for all nodes \(\sigma \in \Sigma\), and (ii) \(\underline{\rho} \leq \underline{\rho}_{i} \) for all \(i \in \Sigma \times \mathcal{H}\).

If the equilibrium allocation is not \(q^*\)-constrained CPO then there exists a sub-tree \(\Gamma_{\tilde{\sigma}}\), with \(\tilde{t} := t(\tilde{\sigma}) \geq 1\), a function \(\tilde{\Delta} : (\Sigma_1 \times \{o\}) \cup \Sigma \to \mathbb{R}^{J+K+1}\), and real numbers \(\bar{P} > 0\) and \(B > 0\), such that

(a) \(\tilde{\Delta} \theta_\delta(\sigma) = 0\), \(-1 \leq \tilde{\Delta} \theta_\delta(\sigma) \leq 1\), and \(-1 \leq \tilde{\Delta} \theta_m(\sigma) \leq 1\),

(b) \(0 < P_\Delta(\sigma) \leq \bar{P}\) and \(0 < C_\Delta(\sigma)\) if \(\sigma \in \Gamma_{\tilde{\sigma}}\),

\[-\bar{P} \leq P_\Delta(\sigma) \leq 0\] for every \(\sigma\) such that \(\sigma_{-1} \in \Gamma_{\tilde{\sigma}}\) and \(\sigma \notin \Gamma_{\tilde{\sigma}}\).

---

\(^{12}\)Precise statements of the definitions of the bounds on the curvature of an agent’s indifference surface, and sufficient conditions under which they are well defined, are notationally heavy; for the case in which markets are assumed to be sequentially complete, a detailed description can be found in Chattopadhyay and Gottardi (1999 Definitions 4 and 5 and Lemma 1). Given our assumption of differentiable strict quasi-concavity and smoothness of the utility function, a parallel development can be carried out for the case of asset markets which is why we do not formalize the concepts.

\(^{13}\)For a vector \(x \in \mathbb{R}^N\), \(\|x\| := (\sum_{i=1}^{N} x_i^2)^{1/2}\), the usual Euclidean norm.
THEOREM 2 (Necessity): Let \( (x^*, \theta^*, q^*, r^*) \) be a competitive equilibrium with a sequence of markets (CE-S) and suppose Assumption 1 holds. Assume that \( x_i^* \in R_{+}^{1+\epsilon} \) for all \( i \in \Sigma \times H \), and that there are real numbers \( \epsilon > 0 \) and \( \rho > 0 \), and a sub-tree \( \Gamma_{\tilde{\sigma}} \), with \( \tilde{t} := t(\tilde{\sigma}) \geq 1 \), such that for all nodes \( \sigma \in \Gamma_{\tilde{\sigma}} \) there exists an agent \( h_{\sigma} \in H \) for whom (i) \( \tilde{\rho} \geq \rho_{\sigma,h_{\sigma}} \), (ii) \( x_{\sigma,h_{\sigma}}^* \geq \epsilon \cdot \mathbf{1}_{(1+\epsilon)x_1} \).

If there exist a function \( \Delta \theta : (\Sigma_1 \times \{o\}) \cup \Sigma \to R^{J+K+1} \), and real numbers \( \bar{P} > 0 \) and \( B > 0 \), such that

1. \( \Delta \theta_\delta(\sigma) = 0 \), \( -1 \leq \Delta \theta_d(\sigma) \leq 1 \), and \( -1 \leq \Delta \theta_m(\sigma) \leq 1 \),
2. \( 0 < P_{\Delta}(\sigma) \leq \bar{P} \) and \( 0 < C_{\Delta}(\sigma) \) if \( \sigma \in \Gamma_{\tilde{\sigma}} \),
3. \( -\bar{P} \leq P_{\Delta}(\sigma) \leq 0 \) for every \( \sigma \) such that \( \sigma^{-1} \in \Gamma_{\tilde{\sigma}} \) but \( \sigma \notin \Gamma_{\tilde{\sigma}} \),
4. for every path \( \sigma^*(\Gamma_{\tilde{\sigma}}) \) in the sub-tree:

\[
\Pi_{\tau=\tilde{t}}^t \left[ \frac{P_{\Delta}(\sigma_{\tau})}{C_{\Delta}(\sigma_{\tau})} \right] \leq 1 \quad \text{for all } t \geq \tilde{t}, \quad \text{and} \quad \lim_{T \to \infty} \sum_{t=\tilde{t}}^T \left\{ \Pi_{\tau=\tilde{t}}^t \left[ \frac{P_{\Delta}(\sigma_{\tau})}{C_{\Delta}(\sigma_{\tau})} \right] \right\} C_{\Delta}(\sigma_t) \leq B,
\]

then the equilibrium allocation is not \( q^* \)-constrained CPO.

REMARK 4: Assets in zero net supply do not play any role in the criterion that we obtain (see the first part of the condition labelled (a) in each result) since they do not play any role in determining the possibilities for intertemporal reassignments; if there are no assets of infinite maturity in non-zero net supply, then all equilibria are necessarily \( q^* \)-constrained CPO. However, every non-redundant asset is important in that it contributes towards determining the equilibrium; in particular, enough inside assets can lead to a sequentially complete market (a case in which our result applies) where the only source of inefficiency is the infinite horizon. But given an equilibrium, the existence of intertemporal improvements is determined without reference to the assets in zero net supply.

From here onwards by the usual assumptions we will mean that Assumption 1 holds, the aggregate endowment is uniformly bounded above across nodes, the allocation is uniformly interior for all agents in all coordinates, and the curvature of every agent’s upper contour set lies in a compact subset of the strictly positive real numbers. These assumptions are easy to state and can be expected to hold in applications even though they are much stronger than the ones under which Theorem 1 or Theorem 2 holds.

The characterization result can be paraphrased as follows: Under the usual assumptions, a \( q^* \)-constrained improvement over an equilibrium allocation exists if and only if there is (a1) a set of nodes which form a sub-tree, denoted \( \Gamma_{\tilde{\sigma}} \), and (a2) a portfolio, denoted \( \Delta \theta \) or \( \Delta \theta \), which is the average variation on the equilibrium asset allocation as per
(i) in Definition 4, with the property that (b) for every node in $\Gamma_{\tilde{\sigma}}$, a node which is its immediate successor is also included in $\Gamma_{\tilde{\sigma}}$ if and only if the portfolio has a positive payoff at that successor node (in particular, there is at least one such successor node) and the payoff from the portfolio is uniformly bounded across $\sigma$ such that $\sigma_{-1} \in \Gamma_{\tilde{\sigma}}$, and (c1) the random discount factors, determined by the product along nodes of the return on the portfolio at a node (where the return at a node is specified by dividing the payoff from the portfolio carried from the previous node by the value of the portfolio at the node, denoted $P_{\Delta}/C_{\Delta}$), are strictly positive and uniformly bounded across nodes in $\Gamma_{\tilde{\sigma}}$ at which they are evaluated, and (c2) the value of the portfolio, calculated by considering the sum of the price at which the portfolio can be bought discounted by the random discount factors mentioned above, is strictly positive, converges along the path, and is uniformly bounded across paths in $\Gamma_{\tilde{\sigma}}$.

The essence of the proof consists in showing that an improvement exists if and only if there exists a set of nodes such that the per capita value of the net transfer to the agents of a generation increases at a quadratic rate as we move to successive generations; this happens since preferences are assumed to be strictly convex and monotone. Since we work with asset markets, and the improvement is restricted to be one that can be obtained via payoffs to existing assets, we are able to use the no arbitrage property of asset prices to write the values of the net transfers in terms of asset prices and the portfolio $\Delta\theta$, or $\Delta\theta$. A complication arises due to the fact that asset payoffs are unrestricted in sign, so that for a given portfolio a node could have successors at which the payoff is positive and other successors at which the payoff is negative; we need to separate these nodes and do so by showing that for the portfolio $\Delta\theta$, or $\Delta\theta$, the ones with positive payoff have the structure of a sub-tree. In addition, because asset payoffs are unrestricted in sign, in Theorem 2 we need to work harder to guarantee that the allocation that is constructed leaves every agent with a vector in his consumption set which is why we require that the equilibrium consumption vector be uniformly interior in every coordinate. Finally, one uses the fact that the payoff of the portfolio $\Delta\theta$, or $\Delta\theta$, is uniformly bounded to obtain the boundedness of the discount factors and the convergence of the family of sums stated as condition (c) in the theorems as a necessary and sufficient condition for the existence of an improvement.

In the skeleton of the argument one finds the seminal proof in Cass (1972). One expects some extension of the Cass result to an environment with uncertainty to hold. Since we work with asset payoffs that are unrestricted in sign and a constrained notion of an improvement, the extension is neither obvious nor simply a matter of cranking through in a mechanical manner. The uniform boundedness of the random discount factors is an entirely new element that shows up. When we compare our results with earlier results on the Cass criterion we see that a very important difference is that our conditions for necessity are stronger since we require that the payoff of the portfolio be

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14Chattopadhyay and Gottardi (1999) provide a discussion and a detailed bibliography of the many refinements to Cass’ proof.
uniformly bounded (which can be ensured by assuming that the aggregate endowment is uniformly bounded) and that consumption is uniformly interior in every coordinate; we need these conditions since asset payoffs are unrestricted in sign. Also, the no arbitrage property of asset prices plays a very important role. The proofs that we give are self-explanatory because of which we prefer not to discuss them in the main text.

REMARK 5: Since the portfolio $\Delta \theta$, or $\Delta \theta$, is uniformly bounded, one can also consider an alternative formulation of the result in which one uses normalized versions of the functions $P_\Delta$ and $C_\Delta$, obtained by dividing them by the norm of the portfolio, and one restricts their domain of definition to the set of nodes on which the norm is positive, i.e., to the sub-tree that is identified; this manoeuvre only changes the values of the constants which give the bounds on the series that appear in Theorems 1 and 2. The reformulation is particularly useful when one deals with the case in which there is only one long-lived asset since the normalized portfolio appears with values in the set $\{-1, 1\}$.

5. SOME SPECIAL CASES AND RELATION TO EARLIER WORK

In this section we turn to see what the result has to say in cases of particular interest. Doing so gives us additional insight into the nature of the characterization.

One Asset

Consider first the case in which there is only one dividend paying asset and no money.

COROLLARY 1: Let $(x^*, \theta^*, q^*, r^*)$ be a competitive equilibrium with a sequence of markets (CE-S) under the usual assumptions. Suppose that $K = 1$ and that fiat money is not available as an asset. The equilibrium allocation is not $q^*$-constrained CPO if and only if there exists a sub-tree, $\Gamma_{\tilde{\sigma}}$ with $\tilde{t} := t(\tilde{\sigma})$, a function $\Delta \theta : (\Sigma_1 \times \{o\}) \cup \Sigma \to R$, and a real number $B > 0$, such that

$$0 < \Delta \theta(\sigma_{-1}) \cdot r_{\tilde{\sigma}}(\sigma) \leq \bar{P} \text{ and } 0 < \Delta \theta(\sigma) \cdot q_{\tilde{\sigma}}(\sigma) \text{ if } \sigma \in \Gamma_{\tilde{\sigma}},$$

$$-\bar{P} \leq \Delta \theta(\sigma_{-1}) \cdot r_{\tilde{\sigma}}(\sigma) \leq 0 \text{ for every } \sigma \text{ such that } \sigma_{-1} \in \Gamma_{\tilde{\sigma}} \text{ and } \sigma \notin \Gamma_{\tilde{\sigma}},$$

and on every path in the sub-tree

$$0 < \Delta \theta(\tilde{\sigma}_{-1}) \cdot \prod_{t=\tilde{t}}^{t} \left[ \frac{r_{\tilde{\sigma}}(\sigma_t)}{q_{\tilde{\sigma}}(\sigma_t)} \right] \cdot \frac{1}{\Delta \theta(\sigma_t)} \leq 1 \text{ for all } t \geq \tilde{t}, \text{ and}$$

$$0 < \lim_{T \to \infty} \Delta \theta(\tilde{\sigma}_{-1}) \cdot \sum_{t=\tilde{t}}^{T} \prod_{\tau=t}^{\tau} \left[ \frac{r_{\tilde{\sigma}}(\sigma_{\tau})}{q_{\tilde{\sigma}}(\sigma_{\tau})} \right] q_{\tilde{\sigma}}(\sigma_t) \leq B.$$

The proof follows directly from the statements of the two theorems.

The result also applies when there is no uncertainty, an instructive case.

The AMSZ result on the net dividend criterion pertains to the case in Corollary 1 with a sequentially complete market. The net dividend criterion ensures that the first of the two conditions fails. It is possible to construct examples in which the net dividend criterion is satisfied but the series in the second condition, the equivalent of the Cass sum,
converges. As our result indicates, in such a case the allocation cannot be improved. The fact that the force of the net dividend criterion could be driving something other than the Cass sum comes out of our characterization in a very clear way.

If the only asset of infinite maturity is fiat money then we have

COROLLARY 2: Let \((x^*, \theta^*, q^*, r^*)\) be a competitive equilibrium with a sequence of markets (CE-S) under the usual assumptions. Suppose that \(K = 0\) but that fiat money is available as an asset. The equilibrium allocation is not \(q^*\)-constrained CPO if and only if there exists a sub-tree, \(\Gamma_\delta\) with \(\bar{t} := t(\bar{\sigma})\), a function \(\bar{\Delta}\theta: (\Sigma_1 \times \{0\}) \cup \Sigma \rightarrow \mathbb{R}\), and a real number \(B > 0\), such that

\[
\begin{align*}
\text{sign } q^*_m(\sigma) &= \text{sign } q^*_m(\bar{\sigma}) \text{ if } \sigma \in \Gamma_\delta, \\
\text{sign } q^*_m(\sigma) &\neq \text{sign } q^*_m(\bar{\sigma}) \text{ for every } \sigma \text{ such that } \sigma_{-1} \in \Gamma_\delta \text{ but } \sigma \notin \Gamma_\delta,
\end{align*}
\]

\[
\text{sign } \bar{\Delta}\theta(\sigma) = \text{sign } \bar{\Delta}\theta(\bar{\sigma}_{-1}) \text{ for all } \sigma \in \Gamma_\delta,
\]

and on every path

\[
0 < \lim_{T \to \infty} \bar{\Delta}\theta(\bar{\sigma}_{-1}) \sum_{t=\bar{t}}^T q^*_m(\sigma_t) \leq B.
\]

PROOF: Since the only long-lived asset is money,

\[
\Pi^T_{t=\bar{t}} \left[ \frac{\mathcal{P}_{\bar{\Delta}}(\sigma_r)}{\mathcal{C}_{\bar{\Delta}}(\sigma_r)} \right] = \Pi^T_{t=\bar{t}} \left[ \frac{\bar{\Delta}\theta(\sigma_{t-1}) \cdot q^*_m(\sigma_t)}{\Delta\theta(\sigma_r) \cdot q^*_m(\sigma_r)} \right] = \frac{\bar{\Delta}\theta(\bar{\sigma}_{-1})}{\Delta\theta(\sigma_{\bar{t}})}
\]

thus showing that the average variation must have the same sign at every node in the sub-tree since the random discount factors are positive. Further, the condition on the series in Theorems 1 and 2 becomes

\[
\lim_{T \to \infty} \sum_{t=\bar{t}}^T \left\{ \Pi^T_{t=\bar{t}} \left[ \frac{\mathcal{P}_{\bar{\Delta}}(\sigma_r)}{\mathcal{C}_{\bar{\Delta}}(\sigma_r)} \right] \right\} \mathcal{C}_{\bar{\Delta}}(\sigma_t) = \lim_{T \to \infty} \sum_{t=\bar{t}}^T \left\{ \frac{\bar{\Delta}\theta(\bar{\sigma}_{-1})}{\Delta\theta(\sigma_{t-1})} \right\} \bar{\Delta}\theta(\sigma_{t-1}) \cdot q^*_m(\sigma_{t-1})
\]

\[
= \lim_{T \to \infty} \bar{\Delta}\theta(\bar{\sigma}_{-1}) \sum_{t=\bar{t}}^T q^*_m(\sigma_{t-1}) \leq B.
\]

Finally, the conditions on the functions \(\mathcal{P}_{\bar{\Delta}}\) and \(\mathcal{C}_{\bar{\Delta}}\) take the form

\[
0 < \bar{\Delta}\theta(\bar{\sigma}_{-1}) \cdot q^*_m(\sigma) \leq \bar{P} \text{ and } 0 < \bar{\Delta}\theta(\sigma) \cdot q^*_m(\sigma) \text{ if } \sigma \in \Gamma_\delta,
\]

\[
-\bar{P} \leq \bar{\Delta}\theta(\bar{\sigma}_{-1}) \cdot q^*_m(\sigma) \leq 0 \text{ for every } \sigma \text{ such that } \sigma_{-1} \in \Gamma_\delta \text{ but } \sigma \notin \Gamma_\delta.
\]

Since we have shown that \(\text{sign } \bar{\Delta}\theta(\sigma) = \text{sign } \bar{\Delta}\theta(\bar{\sigma}_{-1})\) for all \(\sigma \in \Gamma_\delta\), we are led to conclude that

\[
\begin{align*}
\text{sign } q^*_m(\sigma) &= \text{sign } q^*_m(\bar{\sigma}) \text{ if } \sigma \in \Gamma_\delta, \\
\text{sign } q^*_m(\sigma) &\neq \text{sign } q^*_m(\bar{\sigma}) \text{ for every } \sigma \text{ such that } \sigma_{-1} \in \Gamma_\delta \text{ but } \sigma \notin \Gamma_\delta.
\end{align*}
\]

Fiat money is special as a long-lived asset since the condition on the discount factors drops out.
As a special case, under certainty and free disposal of money we have:

COROLLARY 3: Let \((x^*, \theta^*, q^*, r^*)\) be a competitive equilibrium with a sequence of markets (CE-S) under the usual assumptions. Suppose that \(K = 0\) but fiat money is available as an asset. Suppose further that there is no uncertainty, \(S = 1\), and that money is freely disposable. The equilibrium allocation is not \(q^*\)-constrained CPO if and only if there exists a real number \(A > 0\), such that

\[
0 < \lim_{T \to \infty} \sum_{t=i}^{T} q_t^* \leq A := \frac{B}{\Delta \theta(\bar{\sigma}_{-1})}.
\]

PROOF: By free disposal the price of money is non-negative. The condition in Corollary 2 stating that the series is positive implies that \(\Delta \theta(\bar{\sigma}_{-1}) > 0\). Now the result is a restatement of Corollary 2.

In Corollary 3 we recover the classical Cass criterion for one-good deterministic OLG economies (see Okuno and Zilcha (1980) and Balasko and Shell (1980)) since the discounted price of the commodity can be identified with the reciprocal of the price of money. Furthermore, it shows that our characterization result applies to economies with sequentially complete markets and that in such economies all equilibrium allocations need not always be \(q^*\)-constrained CPO.

**Free Disposal of Assets**

An extremely important special case is the one in which assets are freely disposable and dividends are non-negative. Corollary 3 has shown that one cannot rule out the possibility of inefficiency even with free disposability of assets if fiat money is the only asset. In the next result we assume that fiat money is not available; we conjecture that the same result holds even if fiat money were to be available but have not been able to prove it.

PROPOSITION 1: Let \((x^*, \theta^*, q^*, r^*)\) be a competitive equilibrium with a sequence of markets (CE-S) under the usual assumptions. Suppose that fiat money is not available as an asset, that dividends are always non-negative and that assets are freely disposable, \(d : \Sigma \to R^+_K\) and \(q^* : \Sigma \to R^+_K\). If for every path, \(\sigma^\infty\), and every node in the path, \(\sigma_t^\infty\), there exists another node in the path, \(\sigma_s^\infty\), with \(s > t\), and \(d^k(\sigma_s^\infty) > 0\) for all \(k \in K\), then the equilibrium allocation is \(q^*\)-constrained CPO.

PROOF: We apply Theorem 1 using the facts that the asset is freely disposable and that dividends are non-negative, \(q^*(\sigma) \geq 0\) and \(d(\sigma) \geq 0\). By (b) in Theorem 1, \(\Delta d(\sigma) \geq 0\). The additional condition in Proposition 1 ensures that the dividends never stop though they could become arbitrarily small. That ensures that, as in the net dividend criterion, the condition on the discount factors is violated on every sub-tree.
Stationary Equilibrium

We turn to the case in which the equilibrium is stationary which is of particular interest. We will provide results which are analogues of the results in Corollaries 1 and 2. We assume that the economy is stationary, i.e., that the characteristics (endowments and utility functions) of each agent only depend on the realizations of the state during her lifetime, not on time nor on past realizations. Elements of $S$ can now be interpreted as the realizations of a time homogeneous Markov process. Given a node $\sigma$ let $s(\sigma)$ denote the realization of the Markov state. Stationarity of the environment requires that the characteristics of the agents born at a node $\sigma$ and the payoffs of the assets at a node $\sigma$ depend only on $s(\sigma) \in S$. This lets us convert the general model of Section 2 into a stationary one.

**DEFINITION 8:** An economy is stationary if for all $(\sigma, \hat{\sigma}) \in \Sigma \times \Sigma$, $s(\sigma) = s(\hat{\sigma})$ implies that $X_{\sigma,h} = X_{\hat{\sigma},h} := X_{s(\sigma),h}$, $\omega_{\sigma,h} = \omega_{\hat{\sigma},h} := \omega_{s(\sigma),h}$, $u_{\sigma,h} = u_{\hat{\sigma},h} := u_{s(\sigma),h}$, $s(\sigma) = s(\hat{\sigma}) := s_{s(\sigma)}$ and $d(\sigma) = d(\hat{\sigma}) := d_{s(\sigma)}$.

Under stationarity, asset returns will be denoted $(d_{s})_{s \in S} \in R^{S}$ since we will have either one or no long-lived dividend paying asset. Stationary prices of the assets will be denoted $q_{s} := (q_{s,s}, q_{d,s}, q_{m,s})$, $s \in S$.

Stationarity of the equilibrium requires that, given stationary prices, $x_{\sigma,h} = x_{s(\sigma),h}$ for all $(\sigma, h) \in \Sigma \times H$ (i.e., the consumption allocation of each agent only depends on the state at the date of his birth and not on the past). Stationary asset demands will be denoted by $\theta(s, h)$ where $(s, h) \in S \times H$.

**DEFINITION 9 (SCE-S):** $(x^{\ast}, \theta^{\ast}, (q_{1}^{\ast}, \cdots, q_{S}^{\ast}), (r_{1}^{\ast}, \cdots, r_{S}^{\ast}))$ is a stationary competitive equilibrium with a sequence of markets (SCE-S) if it is a CE-S such that for all $(\sigma, \hat{\sigma}) \in \Sigma \times \Sigma$, if $s(\sigma) = s(\hat{\sigma})$ then $x_{\sigma,h} = x_{\hat{\sigma},h}$ and $\theta(\sigma, h) = \theta(\hat{\sigma}, h)$.

**REMARK 6:** We make no claims regarding existence; results on existence are available in certain special cases, e.g., when the only long-lived asset is money.

**PROPOSITION 2:** Let $(x^{\ast}, \theta^{\ast}, (q_{1}^{\ast}, \cdots, q_{S}^{\ast}), (r_{1}^{\ast}, \cdots, r_{S}^{\ast}))$ be a stationary competitive equilibrium with a sequence of markets (SCE-S) and suppose Assumption 1 holds. Suppose that $K = 1$ and that fiat money is not available as an asset. An interior equilibrium allocation is not $q^{\ast}$-constrained CPO if and only if the set $\bar{S} := \{s \in S : 0 < r_{d,s}^{\ast}/q_{d,s}^{\ast} < 1\}$ is non-empty, $\bar{S} \neq \emptyset$.

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15We also need to specify the characteristics and behaviour of the initial old. The way in which this is done is important for questions which deal with existence of equilibrium. Since our interest is in optimality, we shall be sloppy and forego a full specification of the model.

16In Propositions 2 and 3 we no longer need to add “under the usual assumptions” since stationarity of the equilibrium together with Assumption 1 and interior consumption guarantee that the usual assumptions hold.
PROOF: To prove Proposition 2, we invoke Corollary 1. If the allocation is not \( q^* \)-constrained CPO then there exists a sub-tree on which the discount factors are uniformly bounded and a family of sums converges. It follows that there are Markov states for which \( 0 < r_{d,s}^*/q_{d,s}^* \leq 1 \) in order to guarantee that the discount factors are uniformly bounded by 1. In order to guarantee that the series converges, the second inequality must be strict. This provides a proof in one direction.

Going in the other direction, it is evident that \( S \neq \emptyset \) and stationarity of prices implies that the conditions for constructing an improvement are met. ■

Consider replacing the dividend paying asset in Proposition 2 with money. The condition for obtaining a stationary reallocation which also improves can no longer be satisfied (\( S \) is empty since money does not pay a dividend). It is easy to show that now the first order effect of a stationary reallocation of money is zero. So we find that all stationary monetary equilibrium allocations are optimal, i.e., Corollary 3 on monetary equilibria does not extend to the case of stationary equilibrium.

**PROPOSITION 3:** Let \((x^*, \theta^*, (q_1^*, \cdots, q_S^*), (r_1^*, \cdots, r_S^*))\) be a stationary competitive equilibrium with a sequence of markets (SCE-S) and suppose Assumption 1 holds. Suppose that \( K = 0 \) but fiat money is available as an asset. If the allocation is interior it is \( q^* \)-constrained CPO.

**PROOF:** We invoke Corollary 2 applied to a stationary equilibrium. Suppose that the allocation is not optimal. Since \( \text{sign} q_m(\sigma) = \text{sign} q_m(\tilde{\sigma}) \) if \( \sigma \in \Gamma_{\tilde{\sigma}} \), and the number of Markov states is finite, stationarity of prices implies that the series in Corollary 2 cannot converge. This contradicts an implication of the hypothesis that the allocation is not optimal. ■

The result in Proposition 3 brings out in a stark manner the effect of using our notion of constrained optimality. It is easy to show that if the price of money always has the same sign then no CPO improvements exist (restricted or otherwise). Unrestricted improvements do exist when the price of money does not have the same sign in all states. But a single variation does not give the planner the flexibility required to obtain the improvement. In other words, the lack of other long-lived assets is a severe constraint.\(^{17}\)

We make a final comment. It is known that when markets are sequentially complete, i.e., \( J + K + 1 \geq S \) and the assets span \( R^S \), a stationary equilibrium at which the agents’ common matrix of marginal rates of intertemporal substitution has a Perron root which is less than or equal to one is CPO.\(^{18}\) One wonders about the existence of a similar relationship when markets fail to be sequentially complete. We refer the reader to Chattopadhyay

\(^{17}\)This is how we reconcile our result with the claim in Gottardi (1996) regarding the suboptimality of stationary monetary equilibria with money prices changing signs since there an unrestricted improvement does exist but it cannot be implemented given the equilibrium returns on holding money.

\(^{18}\)Very different proofs of the same result can be found in Chattopadhyay and Gottardi (1999), Demange and Laroque (1999), Chattopadhyay (2001).
and Jimenez (2000) who show that it is indeed possible to obtain such a unit root type result even with incompleteness if we assume that dividends are non-negative.

**Arrow Prices**

Since our characterization is based on asset prices, we can use the no arbitrage property to write our results in terms of any Arrow price process. In general there will be many such processes associated with a given equilibrium (unless the market is sequentially complete). The result takes the form of a series in which the terms are weighted reciprocals of the Arrow prices where the weights are given by the product of the payoff to the variation at a node relative to the payoff at all the other nodes which are immediate successors of the same immediate predecessor. Of course, if the allocation is constrained inefficient, the series will converge for all Arrow price processes. We do not explicitly develop the criterion using Arrow prices since we feel that in applications it is much easier to use the criterion in terms of asset prices which can also be related to interest rates.

**Sequentially Complete Markets**

Evidently, our characterization result applies when markets are sequentially complete so that agents can insure against all risks that arise after their birth. We now show that our result covers a wider range of situations relative to existing results on optimality with sequentially complete markets.

Proposition 1 when applied to an economy with sequentially complete markets gives us a condition which is substantially weaker than nonnegligibility. This is because our proof is based on Theorems 1 and 2 which takes into account the second order effects on utility induced by the reallocation.

More generally, the result obtained by Chattopadhyay and Gottardi (1999)—henceforth, CG—can be used to determine whether an equilibrium allocation is CPO when markets are sequentially complete; this can be done by constructing the contingent claims prices from asset prices. However, there is an important caveat. There could exist an equilibrium with a sequence of markets which cannot be represented as an equilibrium in contingent commodity markets. This happens when the dividend paying asset is not freely disposable. Brock (1990) gives a robust deterministic example of such a phenomenon; extensions to stochastic enviroments are easy to obtain. Since CG restrict attention to allocations that can be obtained via trade in ex-ante contingent commodities and their agents have endowments only during their lifetimes, their results cannot be applied; however, our result allows us to determine the optimality properties of the allocation in such cases (in the specific case of Brock’s example the allocation is inefficient).

Furthermore, the characterization result by CG is in terms of a set of weights assigned to different nodes. The weights have no clear interpretation in their work. Here we can relate them to asset returns by writing the criterion using Arrow prices. When there is

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19The contingent claims prices are unique upto normalization, when markets are sequentially complete (see, e.g., Santos and Woodford (1997)).
only one long-lived asset, the weights are completely pinned down and there is no further flexibility. This lack of flexibility indicates the special role of long-lived assets in actually permitting the planner to design a tax/subsidy system which can Pareto improve. In CG the weights have more degrees of freedom since, implicitly, they let the planner use any net intergenerational transfer scheme which respects aggregate resource feasibility without any constraint on how the transfers are to be achieved.

*The Net Dividend Criterion*

Our result provides a complete picture for the case that AMSZ treated. $q^*$-constrained CPO is exactly the notion used by AMSZ though they set things up differently. But there is a difference in that they restrict attention to the market portfolio and, implicitly, to sequentially complete markets. In our result the average variation could be any portfolio (bounded in absolute value) which allows for the possibility that even though no improvements are possible using the market portfolio and returns to it, improvements might exist by using other portfolios or even only one asset which behaves badly but is held forever in equilibrium.

We already noted that the uniform bound on the discount factor is closely related to the purpose for which AMSZ propose the net dividend criterion.

*Empirical Applications*

Starting with Cass (1972), the idea of developing criteria to identify long run inefficiencies has always been tied to their usefulness in applications. The success of the net dividend criterion has to do with the fact that it is easy to use. In fact, AMSZ argued that the US economy does satisfy it and hence it is $q^*$-constrained CPO (in our terminology).

Barbie, Hagedorn, and Kaul (2002) provide a very interesting application of the criterion developed by CG. They write out a particular set of weights with which the criterion can be written in terms of the riskless interest rate. Then they make an assumption of ergodicity and show that data on US government securities indicates that the relevant series converge. They conclude that the allocation of resources in the US economy is not CPO.

Our work clarifies the extent to which both AMSZ and Barbie et al can be correct: as we noted, AMSZ effectively use a constrained notion of optimality and hence it need not be a surprise that they claim optimality while Barbie et al, using an unconstrained notion, claim suboptimality.

Moreover, our work opens the door for further empirical investigation since we provide results which can be applied to different assets separately, even when the asset in question is money. In any empirical work one would have to bear in mind that our asset prices are "real" since we normalize the price of the commodity to one at every date event, i.e., one would have to deflate prices and dividends appropriately before applying the criterion but that is easily done. But it does leave open the possibility that even though the market portfolio cannot be used to obtain a $q^*$-constrained CPO improvement, using some other
asset in isolation or another portfolio does do the job.

6. PROOFS

We introduce some notational conventions and concepts that will be used throughout. For each proof, we will consider an equilibrium tuple \((x^*, \theta^*, q^*, r^*)\) which will remain fixed. We will also use an alternative tuple, denoted \((\widehat{x}, \theta^* + \Delta \theta, q^*, r^*)\), which will be a local variation of \((x^*, \theta^*, q^*, r^*)\) where \(\Delta \theta\) is the variation that makes \(\widehat{x}\) \(q^*\)-constrained feasible, i.e., according to Definition 4.

For an agent \(i\), let \(\Delta x_i := \widehat{x}_i - x^*_i\).

From the budget constraints and monotonicity of preferences we have, for agent \(i\)

\[ x^*_i(\sigma) = \omega_i(\sigma) + \theta^*(i) \cdot r^*(\sigma) \quad \text{for all } \sigma \in \sigma(i)^+. \]

Consequently, for agent \(i\) and for all \(\sigma \in \sigma(i)^+\),

\[ \Delta x_i(\sigma) = \Delta \theta(i) \cdot r^*(\sigma). \]  \(1\)

Let us define the aggregate quantities

\[ \overline{\Delta} x(\sigma) := (1/H) \sum_{h \in \mathcal{H}} \Delta x_{\sigma,h}(\sigma), \]

\[ \overline{\Delta} \theta^a(\sigma) := (1/H) \sum_{h \in \mathcal{H}} \Delta \theta^a(\sigma, h) \quad \text{for } a \in A, \]

\[ \overline{\Delta} \theta(\sigma) := ((\overline{\Delta} \theta^a(\sigma))_{a \in A}). \]

Aggregate feasibility of \(x^*\), Definition 1, and of \(\widehat{x}\), Definition 4, and the fact that, given monotonicity of preferences, the aggregate feasibility constraint holds with equality in an equilibrium, implies that

\[ \overline{\Delta} x(\sigma) + (1/H) \sum_{h \in \mathcal{H}} \Delta x_{\sigma_{-1},h}(\sigma) \leq 0 \]

which, upon substituting (1), leads to

\[ \overline{\Delta} x(\sigma) + \overline{\Delta} \theta(\sigma_{-1}) \cdot r^*(\sigma) \leq 0. \]  \(2\)

We recall the notation for the payoff from a function and its cost, \(P_f\) and \(C_f\). The function we will consider is the average of the variations used to induce the lump sum transfers, \(\overline{\Delta} \theta : (\Sigma_1 \times \{o\}) \cup \Sigma \rightarrow R^{J+K+1}\). So the functions \(P_{\overline{\Delta}} : \Sigma \rightarrow R\) and \(C_{\overline{\Delta}} : \Sigma \rightarrow R\) are given by

\[ P_{\overline{\Delta}}(\sigma) := \overline{\Delta} \theta(\sigma, o) \cdot r^*(\sigma) \quad \text{for all } \sigma \in \Sigma_1 \]

\[ P_{\overline{\Delta}}(\sigma) := \overline{\Delta} \theta(\sigma_{-1}) \cdot r^*(\sigma) \quad \text{for all } \sigma \in \cup_{t \geq 2} \Sigma_t \]

\[ C_{\overline{\Delta}}(\sigma) := \overline{\Delta} \theta(\sigma) \cdot q^*(\sigma) \quad \text{for all } \sigma \in \Sigma. \]
So the feasibility condition (2) takes the form
\[ \Delta x(\sigma) + P_{\Delta}(\sigma) \leq 0. \] (3)

We turn to notation for marginal utility comparisons.

For \( f : \mathbb{R}_+^N \to \mathbb{R} \), \( \partial f(\bar{x}) / \partial x_i \) denotes the partial derivative of the function \( f \) with respect to its \( i \)-th coordinate evaluated at the point \( \bar{x} \).

All derivatives will be evaluated at the chosen equilibrium tuple \( (x^*, \theta^*, q^*, r^*) \); hence, notation for the allocation being considered will be suppressed. For an agent \( i \), \( \partial u_i / \partial x_{\sigma(i)} \) denotes the marginal utility from consumption at the node \( \sigma \in \Sigma_i \). \( \lambda_i(\sigma) \) denotes the Lagrange multiplier on the budget constraint faced by agent \( i \) at the node \( \sigma \in \Sigma_i \).

The first order necessary and sufficient conditions for optimization on the part of agent \( i \) at an interior equilibrium allocation are given by:
\[ \frac{\partial u_i}{\partial x_{\sigma(i)}} = \lambda_i(\sigma) \quad \text{for all } \sigma \in \Sigma_i, \] (4)
\[ \lambda_i(\sigma(i)) \cdot q^a(\sigma(i)) = \sum_{\sigma \in \sigma(i)^+} \lambda_i(\sigma) \cdot r^a(\sigma) \quad \text{for all } a \in A. \] (5)

In particular, equilibrium asset prices satisfy the no arbitrage property of asset prices so that at any given node \( \sigma \in \Gamma \), there exists a vector \( a_\sigma \in \mathbb{R}_+^S \) such that
\[ q^*(\sigma) = \sum_{\sigma' \in \sigma^+} a_\sigma(\sigma') \cdot r^*(\sigma'). \] (6)

With these preliminaries in place, we proceed to the proofs of the various results.

PROOF OF THEOREM 1: Since the allocation is not \( q^* \)-constrained CPO, a constrained improvement must exist; denote it \( \hat{x} \) supported by the variation \( \Delta \theta \). Since both \( x^* \) and \( \hat{x} \) are \( q^* \)-feasible, so is any convex combination of the two, so that without loss of generality we can assume that \( \hat{x} \) is a local variation of \( x^* \).

Consider an agent \( (\sigma, h) \). The change in her utility, up to first order, because of the change in the allocation, is given by
\[ du_i = \frac{\partial u_i}{\partial x_{\sigma(i)}} \cdot \Delta x_i(\sigma(i)) + \sum_{\sigma \in \sigma(i)^+} \frac{\partial u_i}{\partial x_{\sigma(i)^+}} \cdot \Delta x_i(\sigma) \]
which, upon using (4) and (1), can be written as
\[ du_i = \lambda_i(\sigma(i)) \cdot \Delta x_i(\sigma(i)) + \sum_{\sigma \in \sigma(i)^+} \lambda_i(\sigma) \cdot \{ \Delta \theta(i) \cdot r^*(\sigma) \}. \] (7)

Since we have assumed that the alternative is a CPO improvement, the term in (7) must be non-negative for every agent and strictly positive for some agent who could be the initial old.
By using (5), the first order condition for optimal portfolio choice, we obtain

$$\sum_{\sigma \in \sigma(i)^+} \lambda_i(\sigma) \cdot \Delta \theta(i) \cdot r^*(\sigma) = \lambda_i(\sigma(i)) \cdot \Delta \theta(i) \cdot q^*(\sigma(i)).$$  \hspace{1cm} (8)

Using (8) in (7) gives an evaluation of the change in the utility, up to first order, of an agent \(i\) because of the reallocation:

$$\frac{\partial u_i}{\partial x_{\sigma(i)}} \cdot \left\{ \Delta x_i(\sigma(i)) + \Delta \theta(i) \cdot q^*(\sigma(i)) \right\}.$$ \hspace{1cm} (9)

Strict convexity of preferences implies that we need to take into account the second order effect in order to ensure that we have a weak improvement. So it is not sufficient that the expression above be non-negative for every agent; it must exceed a quadratic term given by

$$\rho_i \cdot \left[ \frac{\partial u_i}{\partial x_{\sigma(i)}} \cdot \Delta x_i(\sigma(i)) \right]^2,$$ \hspace{1cm} (10)

where \(\rho_i\) is the greatest lower bound on the curvature of the upper contour set of agent \(i\) at the competitive allocation. So in order to have an improvement, the following inequality must hold for every agent and must be strict for some agent:

$$\frac{\partial u_i}{\partial x_{\sigma(i)}} \cdot \left[ \Delta x_i(\sigma(i)) + \Delta \theta(i) \cdot q^*(\sigma(i)) \right] \geq \rho_i \cdot \left[ \frac{\partial u_i}{\partial x_{\sigma(i)}} \cdot \Delta x_i(\sigma(i)) \right]^2.$$ \hspace{1cm} (11)

By averaging the inequality in (11) across agents born at the same node, and using Jensen’s Inequality applied to a quadratic function, we obtain

$$\bar{\Delta} x(\sigma) + C_\Delta(\sigma) \geq \rho \cdot [\bar{\Delta} x(\sigma)]^2,$$ \hspace{1cm} (12)

(using the function \(C_\Delta\) that we introduced earlier) since \(\rho_i \geq \rho > 0\) by hypothesis (ii) in Theorem 1 on the existence of a positive lower bound on the curvature of the upper contour sets. If an improvement exists then (12) must hold at every node \(\sigma \in \Sigma\) with a strict inequality at some node.

By the feasibility condition on the variation \(\bar{\Delta} \theta_s(\sigma) = 0\), since the short maturity assets are in zero net supply, \(-1 \leq \bar{\Delta} \theta_d(\sigma) \leq 1\), and \(-1 \leq \bar{\Delta} \theta_m(\sigma) \leq 1\). This lets us induce a function, the average variation, called a portfolio, denoted \(\bar{\Delta} \theta : (\Sigma_1 \times \{o\}) \cup \Sigma \to R^{j+k+1}\), which has all the properties stated in the theorem.

Recall the feasibility condition (3)

$$\bar{\Delta} x(\sigma) + P_\Delta(\sigma) \leq 0.$$ \hspace{1cm} (13)

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\(20\)See Definition 4 and Lemma 1 in Chattopadhyay and Gottardi (1999) for an explicit derivation of the required quadratic term in a related context. A similar argument can be used here.
To be able to construct the sub-tree that interests us, we will need an implication of the no arbitrage property of asset prices. Given a node $\sigma \in \Sigma$, let $A(\sigma)$ denote the set of immediate successors at which the payoff from the portfolio is positive, i.e.,

$$A(\sigma) := \{ \sigma' \in \mathcal{P} : \mathcal{P}(\sigma') > 0 \}.$$  

Using the no arbitrage property of asset prices, (6), we have that for every node $\sigma \in \Sigma$ there is $a_\sigma \in R^S_+$ such that

$$C_\Delta(\sigma) = a_\sigma \cdot (\mathcal{P}(\sigma'))_{\sigma' \in A(\sigma)}.$$  \hspace{1cm} (14)

A direct implication of (14) is that if $C_\Delta(\sigma) > 0$ then $A(\sigma) \neq \emptyset$.

We now identify the root of the sub-tree that interests us. First we show that $\mathcal{P}_\Delta(\sigma) = 0$ for all $\sigma \in \Sigma$ cannot hold. If $\mathcal{P}_\Delta(\sigma) = 0$ for all $\sigma \in \Sigma$ then, by (13), $\bar{\Delta} x(\sigma) \leq 0$ for all $\sigma \in \Sigma$, while, by (14), $C_\Delta(\sigma) = 0$ for all $\sigma \in \Sigma$, so that (12) can never hold with a strict inequality contradicting the existence of an improvement.

Let $\bar{\sigma}$ be such that $\mathcal{P}_\Delta(\bar{\sigma}) \neq 0$ and $\mathcal{P}_\Delta(\sigma) = 0$ for all $\sigma$ such that $t(\sigma) < t(\bar{\sigma})$. In the partial order of dates, $\bar{\sigma}$ is the “first node” (it need not be the unique node with this property) with non-zero aggregate transfer to the old. We proceed to verify the existence of a node $\bar{\sigma} \in \bar{\sigma}^+_\theta$ with the additional property that $\mathcal{P}_\Delta(\bar{\sigma}) > 0$. By the definition of $\bar{\sigma}$, $\mathcal{P}_\Delta(\bar{\sigma}^-) = 0$ so that, by (13), $\bar{\Delta} x(\bar{\sigma}^-) \leq 0$; but then, by (12), $C_\Delta(\bar{\sigma}^-) > 0$ necessarily. In case $\mathcal{P}_\Delta(\sigma) \leq 0$ for all $\sigma \in \bar{\sigma}^+_\theta$, with a strict inequality at some node, i.e., the aggregate transfer to the old born at $\bar{\sigma}^-\theta$ is never positive and is negative at some node, then, by (14), $C_\Delta(\bar{\sigma}^-) < 0$ contradicting $C_\Delta(\bar{\sigma}^- \theta) \geq 0$. So, in order to have an improvement $\mathcal{P}_\Delta(\sigma) > 0$ for some $\sigma \in \bar{\sigma}^+_\theta$; denote $\bar{\sigma}$ one of the nodes at which $\mathcal{P}_\Delta(\sigma) > 0$ for $\sigma \in \bar{\sigma}^+_\theta$.\footnote{Clearly, there might exist nodes which are successors to the predecessor of $\bar{\sigma}$, i.e., for $\sigma \in \bar{\sigma}^+_\theta$, such that $\mathcal{P}_\Delta(\sigma) < 0$.}

$\bar{\sigma}$, with date $\tilde{t} := t(\bar{\sigma})$, is the root of the sub-tree that interests us. Since $\tau \in \bar{\sigma}^+_\theta$, $\bar{\sigma}^- = \bar{\sigma}^- \theta$; so $C_\Delta(\bar{\sigma}^-) \geq 0$. Since $\mathcal{P}_\Delta(\bar{\sigma}) > 0$, $\bar{\Delta} \theta(\bar{\sigma}^-) \neq 0$. $\bar{\Delta} \theta(\sigma) = 0$ for all $\sigma$ such that $t(\sigma) < t(\bar{\sigma}^-) = t(\bar{\sigma}) - 1$. Of course, $\tilde{t} \geq 1$.\footnote{The argument needs to be changed slightly when $\tilde{t} = 1$. In this case $C_\Delta(\bar{\sigma}^-) \theta$ is not defined so one cannot say that $C_\Delta(\bar{\sigma}^- \theta) \geq 0$. However, monotonicity of the preferences of the initial old guarantees that $\mathcal{P}_\Delta(\sigma) \geq 0$ for all $\sigma \in \Sigma_1$ so that $\mathcal{P}_\Delta(\sigma) \neq 0$ for all $\sigma \in \Sigma_1$ directly implies that $\mathcal{P}_\Delta(\bar{\sigma}) > 0$ for some $\bar{\sigma} \in \Sigma_1$.}

We can now define the sub-tree that interests us.

$$\Gamma(\bar{\sigma}) := \{ \bar{\sigma} \} \cup \mathcal{A}(\bar{\sigma}) \cup_{\sigma \in \mathcal{A}(\bar{\sigma})} A(\sigma) \cup_{\sigma' \in \bigcup_{\sigma \in \mathcal{A}(\bar{\sigma})} A(\sigma)} A(\sigma') \cdots .$$

The definition is recursive and starts by including $\bar{\sigma}$ and the set $A(\bar{\sigma})$.

We have shown that $\mathcal{P}_\Delta(\bar{\sigma}) > 0$ so, by (13), $\bar{\Delta} x(\bar{\sigma}) < 0$; so (12) at the node $\bar{\sigma}$ implies that $C_\Delta(\bar{\sigma}) > 0$. Therefore, $A(\bar{\sigma}) \neq \emptyset$. So $\mathcal{P}_\Delta(\sigma) > 0$ for $\sigma \in A(\bar{\sigma})$ and (13) implies that

$$0 < \mathcal{P}_\Delta(\sigma) \leq -\bar{\Delta} x(\sigma), \quad \text{for all } \sigma \in A(\bar{\sigma}),$$  \hspace{1cm} (15)
which, when substituted into (12), leads to
\[-P_\Delta(\sigma) + C_\Delta(\sigma) \geq \rho \cdot \left[ P_\Delta(\sigma) \right]^2 \quad \text{for all } \sigma \in A(\tilde{\sigma}). \quad (16)\]

By repeating the argument, we see that the sub-tree \( \Gamma_{\tilde{\sigma}} \) has the following properties:
\[-\Delta x(\sigma) \geq P_\Delta(\sigma) > 0 \quad \text{for all } \sigma \in \Gamma_{\tilde{\sigma}}, \quad (17)\]
by (17) and (12)
\[C_\Delta(\sigma) \geq P_\Delta(\sigma) + \rho \cdot \left[ P_\Delta(\sigma) \right]^2 \quad \text{for all } \sigma \in \Gamma_{\tilde{\sigma}}, \quad (18)\]
by (17) and (18)
\[C_\Delta(\sigma) > 0 \quad \text{for all } \sigma \in \Gamma_{\tilde{\sigma}}, \quad (19)\]
by (19) and (14)
\[A(\sigma) \neq \emptyset \quad \text{for all } \sigma \in \Gamma_{\tilde{\sigma}}, \]
which, with (13), implies that (17) holds.

By aggregate feasibility of consumption and non-negativity of consumption when young, the net intergenerational transfer in equilibrium, given by the return on the aggregate asset endowment, satisfies
\[-\omega(\sigma) \leq \omega \cdot r^*(\sigma) \leq \omega(\sigma).\]

Clearly, the inequality above implies that the payoff from the aggregate feasible portfolio \( \Delta \theta \) is bounded,
\[|P_\Delta(\sigma)| \leq \omega(\sigma) \leq \Omega \quad \text{for all } \sigma \in \Gamma \quad (20)\]
using hypothesis (i) in Theorem 2, i.e., that the aggregate endowment of the consumption good is uniformly bounded across nodes. Let \( \bar{P} := \Omega \). So, by (17) and (20), the function \( P_\Delta \) satisfies \( 0 < P_\Delta(\sigma) \leq \bar{P} \) on the set of nodes such that \( \sigma \in \Gamma_{\tilde{\sigma}} \). Furthermore, it is clear that if a node \( \sigma \) satisfies \( \sigma_{-1} \in \Gamma_{\tilde{\sigma}} \) and \( \sigma \notin \Gamma_{\tilde{\sigma}} \), then \( \sigma \notin A(\sigma_{-1}) \) so that \( -\bar{P} \leq P_\Delta(\sigma) \leq 0 \), as stated in (b) of Theorem 2.

By (19), \( 0 < C_\Delta(\sigma) \) on the set \( \Gamma_{\tilde{\sigma}} \) as stated in (b) of Theorem 2.

Since, by (17), \( P_\Delta(\sigma) > 0 \), for any node in the sub-tree \( \Gamma_{\tilde{\sigma}} \) (18) can be rewritten as
\[\frac{1}{C_\Delta(\sigma)} \leq \frac{1}{P_\Delta(\sigma)} - \frac{\rho}{1 + \rho \bar{P}_\Delta(\sigma)} \quad \text{for all } \sigma \in \Gamma_{\tilde{\sigma}}, \quad (21)\]
while \( \rho > 0 \) and (20) imply that
\[\frac{\rho}{1 + \rho \Omega} \leq \frac{\rho}{1 + \rho \bar{P}_\Delta(\sigma)} \quad \text{for all } \sigma \in \Gamma_{\tilde{\sigma}}.\]

Using the above in (21) we obtain the condition
\[\frac{1}{C_\Delta(\sigma)} + \frac{\rho}{1 + \rho \Omega} \leq \frac{1}{P_\Delta(\sigma)} \quad \text{for all } \sigma \in \Gamma_{\tilde{\sigma}},\]
the series, being bounded and increasing, converges. It follows that
\[ \prod_{t=1}^{T-1} \frac{P(\sigma_t)}{C(\sigma_t)} + \frac{\rho}{1 + \rho \Omega} P(\sigma) \leq 1 \]
for all \( \sigma \in \Gamma_{\delta}. \) (22)

By iterating on the inequality (22) along paths in the sub-tree \( \Gamma_{\delta} \) we obtain
\[ \Pi_{t=i}^{T-1} \frac{P(\sigma_t)}{C(\sigma_t)} + \frac{\rho}{1 + \rho \Omega} \sum_{t=i}^{T-1} \left\{ \Pi_{\tau=t}^{t-1} \left[ \frac{P(\sigma_{\tau})}{C(\sigma_{\tau})} \right] \right\} C(\sigma_t) \leq 1 \]
\[ \Rightarrow \Pi_{t=i}^{T-1} \frac{P(\sigma_t)}{C(\sigma_t)} + \frac{\rho}{1 + \rho \Omega} \sum_{t=i}^{T-1} \left\{ \Pi_{\tau=t}^{t-1} \left[ \frac{P(\sigma_{\tau})}{C(\sigma_{\tau})} \right] \right\} C(\sigma_t) \leq 1 \]
\[ \Rightarrow \Pi_{t=i}^{T} \frac{P(\sigma_t)}{C(\sigma_t)} + \frac{\rho}{1 + \rho \Omega} \sum_{t=i}^{T} \left\{ \Pi_{\tau=t}^{t-1} \left[ \frac{P(\sigma_{\tau})}{C(\sigma_{\tau})} \right] \right\} C(\sigma_t) \leq 1. \]

Since all the terms in the inequality above, including each term in the series, are positive, the series, being bounded and increasing, converges. It follows that
\[ \Pi_{\tau=t}^{t-1} \left[ \frac{P(\sigma_{\tau})}{C(\sigma_{\tau})} \right] \leq 1 \]
\[ \lim_{T \to \infty} \sum_{t=i}^{T} \left\{ \Pi_{\tau=t}^{t-1} \left[ \frac{P(\sigma_{\tau})}{C(\sigma_{\tau})} \right] \right\} C(\sigma_t) \leq B := \frac{1}{\rho} + \Omega \]
along the path. The proof is completed by noting that the argument applies to every path in the sub-tree. \( \blacksquare \)

PROOF OF THEOREM 2: We need to construct a feasible allocation, obtainable via a variation of asset holdings, which improves over the equilibrium allocation. We do so by first proposing a candidate variation, then verifying that the induced consumption allocation is aggregate feasible and gives consumption vectors in the consumption set of each agent, and finally verifying that the proposed transfers satisfy a curvature condition which guarantees that we have an improvement.

\( \Gamma_{\delta} \), a sub-tree, denotes the set of nodes that we will work with. From here onwards \( \sigma \in \Gamma_{\delta} \) unless otherwise noted. \( \Delta \theta, C_{\Delta}, P_{\Delta}, \) etc., refer to the functions whose existence is assumed in the statement of Theorem 2.

From now on consider a fixed path in the sub-tree, \( \sigma^\infty(\Gamma_{\delta}) \).

Let
\[ R_{\Delta}(\sigma_t) := \Pi_{t=1}^{t-1} \left[ \frac{P(\sigma_{\tau})}{C(\sigma_{\tau})} \right]. \]

By hypothesis (c) in Theorem 2, \( 0 < R_{\Delta}(\sigma_t) \leq 1 \) and
\[ 0 < \lim_{T \to \infty} \sum_{t=i}^{T} R_{\Delta}(\sigma_t) C(\sigma_t) \leq B. \] (23)

Define
\[ a(\sigma_t) := \frac{\kappa}{1 + \kappa \max\{1, \rho\}} \frac{1}{\max\{1, B^2\}} \cdot R_{\Delta}(\sigma_t) \left[ \sum_{s=t}^{t} R_{\Delta}(\sigma_s) C(\sigma_s) \right] \quad \text{for } \sigma_t \in \Gamma_{\delta}, \]
\[ a(\sigma) := 0 \quad \text{otherwise,} \]

where \( \kappa := (1/2) \frac{\varepsilon}{\bar{P}} \) and \( \bar{\rho} > 0, \varepsilon > 0, \) and \( \bar{P} > 0 \) are specified in hypotheses (i), (ii) and (c) of Theorem 3. Clearly,

\[
\frac{\kappa}{1 + \kappa \max\{1, \bar{\rho}\}} < 1
\]

(24)

\[
\frac{B}{\max\{1, B^2\}} \leq 1.
\]

(25)

So, using \( R_{\Delta}(\sigma_i) \leq 1, \) (23), (24), and (25), we have

\[
0 \leq a(\sigma) \leq \frac{\kappa}{1 + \kappa \max\{1, \bar{\rho}\}} \cdot \frac{1}{\max\{1, B^2\}} \cdot B = \frac{\kappa}{1 + \kappa \max\{1, \bar{\rho}\}} \cdot \frac{B}{\max\{1, B^2\}} < 1.
\]

(26)

The reallocation can now be defined where \( h_\sigma \) is specified in Theorem 2. Consumption when young is reassigned as follows

\[
\hat{x}_{\sigma, h}(\sigma) := x^*_{\sigma, h}(\sigma) + (-1) a(\sigma - 1) \cdot P_{\Delta}(\sigma) \quad \text{for } \sigma \in \Sigma,
\]

while consumption when old is induced according to

\[
\hat{x}_{\sigma, h}(\sigma') := x^*_{\sigma, h}(\sigma') + a(\sigma) \cdot P_{\Delta}(\sigma') \quad \text{for } \sigma' \in \sigma^+, \quad \text{for } \sigma \in \Sigma,
\]

and similarly for the initial old. For \( h \neq h_\sigma, \) \( x^*_i = \hat{x}_i \) so their consumption is unaffected.

Evidently, at each node the change in the consumption allocation of an old agent is offset by an identical change of opposite sign in the consumption allocation of a young agent; thus aggregate feasibility is always maintained by construction. We now show that every individual obtains a consumption vector in his consumption set. For a young agent born at \( \sigma, \) where \( \sigma - 1 \in \Gamma_\sigma, \)

\[
\Delta \hat{x}_{\sigma, h}(\sigma) := \hat{x}_{\sigma, h}(\sigma) - x^*_{\sigma, h}(\sigma) = (-1) a(\sigma - 1) \cdot P_{\Delta}(\sigma)
\]

\[
\Rightarrow \quad \| \Delta \hat{x}_{\sigma, h}(\sigma) \| \leq |a(\sigma)| \cdot \bar{P}
\]

since, by hypothesis (b) of Theorem 2, \( |P_{\Delta}(\sigma)| \leq \bar{P} \) for every \( \sigma \) such that \( \sigma - 1 \in \Gamma_\sigma, \)

\[
\Rightarrow \quad \| \Delta \hat{x}_{\sigma, h}(\sigma) \| \leq \frac{\kappa}{1 + \kappa \max\{1, \bar{\rho}\}} \cdot \frac{B}{\max\{1, B^2\}} \cdot \bar{P}
\]

\[
= \frac{1}{2} \varepsilon \cdot \frac{1}{1 + (1/2)(\varepsilon/\bar{P}) \max\{1, \bar{\rho}\}} \cdot \frac{B}{\max\{1, B^2\}} < \varepsilon
\]

using (25) and the definition of \( \kappa. \) By hypothesis (ii) of Theorem 2, \( \varepsilon > 0 \) is a uniform lower bound on every coordinate of the equilibrium consumption vector, i.e., when young and in all states faced when old. So in states in which the young transfer the good to the old, the post-transfer consumption of the young is strictly positive (and the old consume a positive amount since they receive the transfer); the states in which the young receive the good are the states in which the old are forced to pay up but the bound above is
valid for all states and that implies that the old never surrender more than \( \varepsilon \) which leaves them with strictly positive post-transfer consumption. So every agent gets a vector in the interior of her consumption set.

We have shown that the proposed reallocation is feasible. We proceed to show that we have an improvement.

Consider a node which is a successor to a node in the sub-tree but which is not an element of the sub-tree, \( \sigma_{-1} \in \Gamma_{\sigma} \) but \( \sigma \notin \Gamma_{\sigma} \). In such a case the old agent delivers the commodity to a young agent, so \( \mathcal{P}_\Delta(\sigma) < 0 \) and \( \tilde{x}_{\sigma,h}(\sigma) > x^*_\sigma(\sigma) \), and the young agent’s asset holding is not perturbed so that, by monotonicity of preferences, such a young agent is improved.

For an agent whose asset holding is perturbed a sufficient condition for a local change from the equilibrium allocation to be weakly improving is that the inequality below holds\(^{23}\)

\[
\frac{\partial u_{\sigma,h}}{\partial x_{\sigma}} \cdot \Delta x_{\sigma,h}(\sigma) + \sum_{\sigma' \in \sigma^+} \frac{\partial u_{\sigma,h}}{\partial x_{\sigma'}} \cdot \Delta x_{\sigma,h}(\sigma') \geq \tilde{\rho}_{\sigma,h} \cdot \frac{\left[ \frac{\partial u_{\sigma,h}}{\partial x_{\sigma}} \cdot \Delta x_{\sigma,h}(\sigma) \right]^2}{\frac{\partial u_{\sigma,h}}{\partial x_{\sigma}}}.
\]

Upon using (5) and the values of \( \Delta x_{\sigma,h}(\sigma) \) induced by the proposed reallocation we obtain the inequality

\[
-a(\sigma_{t-1})\mathcal{P}_\Delta(\sigma_t) + a(\sigma_t)\mathcal{C}_\Delta(\sigma_t) \geq \tilde{\rho}_{\sigma,h} \cdot \left[ a(\sigma_{t-1})\mathcal{P}_\Delta(\sigma_t) \right]^2.
\]

We proceed to check that the proposed reallocation does indeed satisfy (27).

\[
a(\sigma_t) - \frac{\mathcal{P}_\Delta(\sigma_t)}{\mathcal{C}_\Delta(\sigma_t)}a(\sigma_{t-1}) = \frac{\kappa}{1 + \kappa \max\{1, \tilde{\rho}\}} \max\{1, B^2\} \left[ \mathcal{R}_\Delta(\sigma_t) \mathcal{R}_\Delta(\sigma_t) \mathcal{C}_\Delta(\sigma_t) \right]
\]

\[
+ \mathcal{R}_\Delta(\sigma_t) \sum_{s=t}^{t-1} \mathcal{R}_\Delta(\sigma_s) \mathcal{C}_\Delta(\sigma_s) - \frac{\mathcal{P}_\Delta(\sigma_t)}{\mathcal{C}_\Delta(\sigma_t)} \mathcal{R}_\Delta(\sigma_t) \sum_{s=t}^{t-1} \mathcal{R}_\Delta(\sigma_s) \mathcal{C}_\Delta(\sigma_s) \right]
\]

\[
= \frac{\kappa}{1 + \kappa \max\{1, \tilde{\rho}\}} \max\{1, B^2\} \left[ \mathcal{R}_\Delta(\sigma_t) \mathcal{R}_\Delta(\sigma_t) \mathcal{C}_\Delta(\sigma_t) \right]
\]

where we substitute for \( a(\sigma) \) and use the fact that

\[
\mathcal{R}_\Delta(\sigma_t) = \Pi_{\tau=t}^t \left[ \frac{\mathcal{P}_\Delta(\sigma_{\tau})}{\mathcal{C}_\Delta(\sigma_{\tau})} \right] = \Pi_{\tau=1}^{t-1} \left[ \frac{\mathcal{P}_\Delta(\sigma_{\tau})}{\mathcal{C}_\Delta(\sigma_{\tau})} \right] \cdot \mathcal{C}_\Delta(\sigma_t) \cdot \mathcal{R}_\Delta(\sigma_t).
\]

So

\[
a(\sigma_t) - \frac{\mathcal{P}_\Delta(\sigma_t)}{\mathcal{C}_\Delta(\sigma_t)} a(\sigma_{t-1}) = \frac{\kappa}{1 + \kappa \max\{1, \tilde{\rho}\}} \max\{1, B^2\} \mathcal{C}_\Delta(\sigma_t) \left[ \mathcal{R}_\Delta(\sigma_t) \right]^2.
\]

Using (23) we have

\[
1 \geq \left[ \frac{\sum_{s=1}^{t-1} \mathcal{R}_\Delta(\sigma_s)\mathcal{C}_\Delta(\sigma_s)}{\max\{1, B\}} \right].
\]

\(^{23}\)See Definition 5 and Lemma 1 in Chattopadhyay and Gottardi (1999) for an explicit derivation of the required quadratic term in a related context. A similar argument can be used here.
Using (24) and (30) we obtain
\[
1 \geq \frac{\kappa \{1, \bar{\rho}\}}{1 + \kappa \{1, \bar{\rho}\}} \left[ \frac{\sum_{s=1}^{t-1} R_{\Delta}(s) C_{\Delta}(s)}{\max\{1, B\}} \right]^2.
\] (31)

So (29) and (31) imply that
\[
a(\sigma_t) - \frac{P_{\Delta}(\sigma_t)}{C_{\Delta}(\sigma_t)} a(\sigma_{t-1}) \geq \{1, \bar{\rho}\} C_{\Delta}(\sigma_t)
\[
\times \left[ \frac{\kappa}{1 + \kappa \{1, \bar{\rho}\}} \frac{1}{\max\{1, B\}} R_{\Delta}(\sigma_t) \sum_{s=1}^{t-1} R_{\Delta}(s) C_{\Delta}(s) \right]^2
\[
= \{1, \bar{\rho}\} C_{\Delta}(\sigma_t) \left[ \frac{P_{\Delta}(\sigma_t)}{C_{\Delta}(\sigma_t)} a(\sigma_{t-1}) \right]^2
\[
\geq \bar{\rho}_{\sigma,h} C_{\Delta}(\sigma_t) \left[ \frac{P_{\Delta}(\sigma_t)}{C_{\Delta}(\sigma_t)} a(\sigma_{t-1}) \right]^2
\]
where we use (28) and the facts that (i) $\max\{1, B\}^2 = \max\{1, B^2\}$ as $B > 0$, (ii) $\max\{1, \bar{\rho}\} \geq \bar{\rho} \geq \bar{\rho}_{\sigma,h}$ for all $\sigma \in \Gamma_{\tilde{\sigma}}$, and $h = h_{\sigma}$. This shows that the proposed reallocation does indeed satisfy (27) so that an agent whose asset holding is perturbed is also at least weakly improved.

Finally, the first agent to receive an asset transfer is one who is born at the node $\tilde{\sigma}$. She does not transfer consumption to those who are old when she is young since, by construction, $a(\tilde{\sigma}_{t-1}) = 0$; however, $a(\tilde{\sigma}) > 0$ by construction and $C_{\Delta}(\tilde{\sigma}) > 0$ by hypothesis (b) of Theorem 2, so for her (27) is satisfied with a strict inequality ensuring that overall we have a strict improvement.

We have verified that the proposed reallocation is an improvement thus completing the proof of the theorem. ■
REFERENCES


