Long-run welfare effects of common property public infrastructure

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Abstract
By means of an overlapping generations model we study some long-run steady state features prompted by a free-access public capital which dissipates its return among the private factors, because it enters a constant-returns-to-scale production function. The public investment is funded by lump-sum taxes in both younger and older generations. Our main conclusion is that, in general, the utility of an individual living in the long run steady state equilibrium may decline, even when the private capital-labor ratio increases, as a consequence in both an increase in per-capita public investment, and a shift of the tax burden from the younger to the older generation, holding the per-capita public investment constant.

Keywords: Public capital, rent dissipation, long-run steady state.
JEL Classification: E69, H49.

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1 Introduction

Economic activity is commonly carried out in a world surrounded by free-access public capital stock or infrastructure, such as roads, streets, bridges, water and sewer treatment systems, and so on. Indeed, classical economists, like Adam Smith, already pointed out the importance of the existence of a public infrastructure for the support of markets.

When public capital enters a constant-returns-to-scale production function there are decreasing returns to scale in the private factors. This is the so-called profit-augmenting or unpaid-factor case, and implies that when private factors are hired at their marginal products, economic profits arise as a consequence of the Euler’s formula. In this trend, the solutions proposed to maintain the equilibrium zero-profit condition have been diverse and range from assuming that the public-capital-profit is fully appropriated by the government (Basu, 1987); is shared by government and consumers (Keen and Marchand, 1997); is charged by a rental on its use (Pestiau, 1974); or is captured by one of the factors (Kellermann, 2007). However, in a recent paper, Feehan and Batina (2007) propose a different solution for this feature by considering a parameter that specifies the share of the public-capital-profit which is dissipated among private factors. This parameter is simply the share of total income accruing to these factors, and its effect is that, at zero profit equilibrium, the prices of private factors are set above their marginal products in such a way that firms over-hire them, making an inefficient use of resources. This situation can be interpreted as a phenomenon of congestion equivalent to the case of a common property resource. Feehan and Batina apply this characterization to deriving the optimal design of the private factor taxes that are employed to finance a public input, by using a static-partial-equilibrium model where both the interest rate and the amount of private capital stock are exogenous.

The goal of this paper is to extend Feehan and Batina’s price rule to a dynamic general equilibrium model where both prices and factor quantities are endogenously determined. For that purpose we are going to consider an overlapping generations model similar to that used by Pestiau (1974), where individuals live two periods and the aggregated linear-homogeneous production function depends on labor, private capital and free-access public capital which is financed by lump-sum taxes in both the younger and older generation, as in Bierwag, Grove and Kang (1969). Our main concern will be to study the long-run steady state effects on private capital stock and welfare of changes in both the weight of the tax burden borne by each generation and the public investment policy. In addition, the adoption of Feehan and Batina’s price rule will allow us to characterize the phenomenon of congestion in an endogenous way, departing from other dynamic general-equilibrium approaches such as that by Glomm and Ravikumar (1994), who tackle the congestion by adjusting the stock of public capital to the aggregate use of private factors, and that by Fisher and Turnovsky (1998), who draw a congestion function from the public good literature to a Ramsey model.

The remainder of this paper is structured as follows. Section 2 presents
the model and the equilibrium concept. Section 3 states our main conclusions in the long-run steady state and, finally, Section 4 includes some final remarks and comments.

2 The model

2.1 Households

Let us consider an overlapping-generations economy where the population on date \( t \geq 1 \) is given by \( L_t \) and it grows at an exogenous rate \( n > -1 \) so that

\[
L_t = (1 + n)L_{t-1},
\]

(1)

with \( L_0 \) given. Each individual lives for two periods. During the first period, when individuals are young, they work, and in the second period, when they are old, they are retired from the labor force. An individual born in period \( t \) is endowed with one unit of labor, which is supplied inelastically and saves during youth in order to consume when he is old.

The representative household is characterized by a two period-separable utility function

\[
u(c_y^t, c_o^{t+1}) = v(c_y^t) + \beta v(c_o^{t+1}),
\]

(2)

where \( c_y^t \) denotes the consumption of the agent at time \( t \) when young, \( c_o^{t+1} \) is the consumption of the agent at time \( t + 1 \) when old, and \( 0 < \beta < 1 \) is the household’s discount factor. Let us assume that \( v' > 0, v'' < 0 \), and that both \( c_y^t \) and \( c_o^{t+1} \) are normal commodities.

The representative household maximizes its utility subject to the intertemporal budget constraints

\[
c_y^t + S_t = W_t - T_y^t,
\]

\[
c_o^{t+1} = (1 + r_{t+1})S_t - T_o^{t+1}
\]

Where \( S_t \) is saving, \( W_t \) is the wage, \( T_y^t \) and \( T_o^{t+1} \) are lump-sum taxes when young and old respectively, and \( r_{t+1} \) is the interest rate. In this trend, the consolidated budget constraint is

\[
c_y^t + \frac{c_o^{t+1}}{1 + r_{t+1}} = W_t - T_y^t - \frac{T_o^{t+1}}{1 + r_{t+1}}
\]

(3)

From the first order condition of the problem of maximizing (2) subject to (3) we hold that

\[(1 + r_{t+1})\beta \frac{\partial v}{\partial c_o^{t+1}} = \frac{\partial v}{\partial c_y^t},
\]

(4)

which yields the solution

\[
e_y^t(W_t, T_y^t, T_o^{t+1}, r_{t+1}), c_o^{t+1}(W_t, T_y^t, T_o^{t+1}, r_{t+1}),
\]

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so that
\[ S_t(W_t, T^y_t, T^o_{t+1}, r_{t+1}) = W_t - T^y_t - c^y_t(W_t, T^y_t, T^o_{t+1}, r_{t+1}). \] (5)

From the previous assumption of normality in consumption in both periods
\[ \frac{\partial c^y_t}{\partial W_t} > 0 \quad \text{and} \quad \frac{\partial c^y_{t+1}}{\partial W_t} > 0, \]
which implies that
\[ \frac{\partial S_t}{\partial W_t} = 1 - \frac{\partial c^y_t}{\partial W_t} > 0. \] (6)

In addition, by writing consumption when young in equilibrium as
\[ c^y_t(m_t, r_{t+1}), \]
where,
\[ m_t = W_t - T^y_t - \frac{1}{1+r_{t+1}} T^o_{t+1}. \]
Exploiting Eq. (5), it is fairly easy to prove
\[ \frac{\partial S_t}{\partial T^y_t} = - \frac{\partial S_t}{\partial W_t} < 0, \]
\[ \frac{\partial S_t}{\partial T^o_{t+1}} = \frac{1}{1+r_{t+1}} \left( 1 - \frac{\partial S_t}{\partial W_t} \right) > 0, \] (7)
a property that will be useful for further results. In turn, regarding the variation of savings with respect to the interest rate, we will assume that
\[ \frac{\partial c^y_t}{\partial r_{t+1}} \leq 0, \]
which means that substitution effect dominates income effect. This assumption will be useful to keep the stability of the model as long as
\[ \frac{\partial S_t}{\partial r_{t+1}} = - \frac{\partial c^y_t}{\partial r_{t+1}} \geq 0 \] (See Galor and Ryder, 1989).

2.2 Aggregated production function
In every period, the private consumption good is produced by a technology that uses three factors: private capital, public capital and labor, denoted by \( K_t, G_t \) and \( L_t \) respectively. The technology displays constant returns to scale and is represented by a production function \( F \) which is homogeneous of degree one in all inputs. That is, an equiproportional increase in the factors increases output of the consumption commodity in the same proportion. As we have remarked in the Introduction, this type of public input is typically referred to as unpaid factor and is subject to congestion, i.e., there are decreasing returns in the private factors. Let \( k_t = K_t/L_t \) and \( g_t = G_t/L_t \) be the private and the public capital-labor ratios, and \( f(k_t, g_t) \) the output-labor ratio so that
\[ f(k_t, g_t) = F(K_t, G_t, L_t)/L_t = F(k_t, g_t, 1). \]

\[ ^1 \text{For a CRRA utility function, } v(c) = \frac{c^{1-\theta}}{1-\theta}, \theta > 0, \text{ this assumption implies that the elasticity of substitution } \varepsilon = \theta^{-1} \text{ has to be higher than one, in particular } \varepsilon > 1. \]
\[ ^2 \frac{T^o_{t+1}}{(W_t-T^y_t)^{\frac{1}{2}}} \text{ excluding the logarithmic case, } \theta = 1. \]
Let us assume that this function is homogeneous of degree \(0 < \alpha < 1\). Therefore, Euler’s formula can be applied

\[
\alpha f(k_t, g_t) = f_k k_t + f_g g_t.
\]

(8)

In addition, since \(f(k_t, g_t)\) exhibits decreasing returns,

\[
f_i > 0, \quad f_{ii} < 0, \quad f_{ij} > 0, \quad i, j = k_t, g_t; \quad i \neq j.
\]

Where \(f_i\) is the partial derivative of \(f\) with respect to \(i = k_t, g_t\). Let us state the following notation for elasticities:

\[
\eta_i = f_i f, \quad i = k_t, g_t.
\]

(9)

\[
\eta_{ij} = f_{ij} f_i f_j, \quad i, j = k_t, g_t.
\]

(10)

It is fairly easy to see that

\[
\eta_i > 0, \quad \eta_{ii} < 0, \quad \eta_{ij} > 0, \quad i, j = k_t, g_t; \quad i \neq j.
\]

On the other hand, by dividing Eq. (8) by \(f\) we hold that

\[
0 < \eta_k + \eta_g = \alpha < 1,
\]

and, partially deriving Eq. (8) with respect to \(k_t\), and dividing by \(f_k\) we have

\[
\eta_{gk} + \eta_k f_k = \alpha - 1 < 0,
\]

and

\[
\eta_{gg} + \eta_g f_g = \alpha - 1 < 0.
\]

(11)

(12)

by making the same operation with respect to \(g_t\).

### 2.3 Prices

Since the aggregated production function is linearly homogeneous and it is assumed that public capital is freely available, to set private factor prices to their marginal products is not an equilibrium since the null profit condition does not hold. In this trend, we follow Feehan and Batina’s (2007) price rule

\[
R_t \equiv \delta_k + \tau_t = F_K + \gamma_t F_G \frac{G_t}{K_t},
\]

(13)

\[
W_t = F_L + (1 - \gamma_t) F_G \frac{G_t}{L_t},
\]

\[\text{2This assumption allow us to obtain a clear-cut sign for the partial derivatives of factor prices and, for instance, it runs with the case of a nested CES-type aggregated production function } F(K_t, G_t, L_t) = (\lambda K_t^\rho + (1 - \lambda)G_t^{(1-\rho)/\rho}L_t^{1-\rho}, 0 < \lambda < 1, 0 < \alpha < 1, \rho \leq 1.\]
where $R_t$ is the rate of return of the private capital, $0 \leq \delta_k \leq 1$ is its depreciation rate, and $0 \leq \gamma_t \leq 1$ is the share of public capital return which goes to private capital return. Note that according to our price rule

$$Y_t = R_t K_t + W_t L_t,$$

This allows us to express wages in per capita terms as

$$W_t = f(k_t, g_t) - R_t k_t.$$  \hspace{1cm} (14)

In addition, Feehan and Batina (2007) define $\gamma_t$ as the share of total income accruing to private capital, that is

$$\gamma_t = \frac{R_t K_t}{Y_t}.$$  \hspace{1cm} (15)

Thus, substituting Eq. (15) in Eq. (13) and rearranging terms, the rate of return of the private capital can be written as,

$$R_t = \frac{F_K Y_t}{Y_t - F_G G_t},$$

which, in per capita (and elasticities) terms, is

$$R_t = \frac{f_k f(k_t, g_t)}{f(k_t, g_t) - f_g g_t} = \frac{f_k}{1 - \eta_g}.$$  \hspace{1cm} (16)

We see that, according to (10), under this price rule the rate of return of private capital is higher than its marginal product for every level of $k$. In addition, since Eq. (16) depends on total product and both marginal products, it is advisable to study how it varies with changes in both private and public capital-labor ratio stock.

**Proposition 1** $\frac{\partial R_t}{\partial g_t} > 0$ and $\frac{\partial R_t}{\partial k_t} < 0$.

Proof: see appendix.

### 2.4 Government

The Government’s budget constraint is given by:

$$I_t^G = T_t L_t + T_{t-1}^o L_{t-1},$$

where $I_t^G$ is public investment on date $t$. Dividing by $L_t$, taking into account (1), we can express the Government’s budget constraint in per capita terms as

$$i_t^g = T_t^g + \frac{T_{t-1}^o}{1 + n}.$$
On the other hand, public investment as such is given by

\[ I_t^G = G_{t+1} - G_t + \delta gG_t, \]

dividing by \( L_t \), taking into account (1), we can express the public investment in per capita terms as

\[ i_t^G = (1 + n)g_{t+1} - (1 - \delta_e)g_t. \]

Substituting public investment, the Government’s budget constraint can be written as

\[ T_t^y + \frac{T_t^o}{1+n} = (1 + n)g_{t+1} - (1 - \delta_e)g_t. \] (17)

In what follows we are going to consider a constant public investment per capita policy so that

\[ g_{t+1} = g_t = g. \]

In addition, as in Bierwag, Grove and Kang (1969), let us define \( T_t^y = (1 - \tau)T_t \) so that \( 0 \leq \tau \leq 1 \) and \( (1 - \tau)T_t + \frac{1}{1+n}T_t^o = T_t \), then \( T_t^o = (1 + n)\tau T_t \), substituting in (17), we have that

\[ T_t^y = (1 - \tau)(\delta g + n)g, \]
\[ T_t^o = (1 + n)\tau(\delta g + n)g, \] (18)

which depend on the rate of reposition of public capital \( \delta g + n \).

### 2.5 Equilibrium and stability

Given initial private and public capital-labor ratios \( (k_0, g_0) \), an equilibrium is a sequence of allocations \( \{c_t^y, c_t^o, k_t, g_t\}_{t=0}^{\infty} \), factor prices \( \{W_t, R_t\}_{t=0}^{\infty} \), and lump sum taxes \( \{T_t^y, T_t^o\}_{t=0}^{\infty} \) such that:

(i) Given the factor prices and the taxes, the allocation solves the maximization problem of each consumer;

(ii) Given the allocation, the factor prices are consistent with the firms’ profit maximization;

(iii) The market for the consumption commodity clears at every date; and

(iv) \( T_t^y + \frac{T_t^o}{1+n} = (1 + n)g_{t+1} - (1 - \delta_e)g_t \) and \( s_t = (1 + n)k_{t+1} - (1 - \delta_e)k_t \).

Notice that the accumulation expression for the capital-labor ratio provided by the notion of equilibrium can be written, taking into account (1), as follows:

\[ S_t(W_t, T_t^y, T_t^o, r_{t+1}) = (1 + n)k_{t+1} - (1 - \delta_e)k_t. \] (19)

As is well known, for the stability of the model it is necessary that

\[ 0 < \frac{dk_{t+1}}{dk_t} < 1. \]

Therefore, taking the total derivative of Eq. (19) with respect to \( k_t \)

\[ \frac{\partial S_t}{\partial W_t} \frac{\partial W_t}{\partial k_t} + \frac{\partial S_t}{\partial r_{t+1}} \frac{\partial r_{t+1}}{\partial k_t} + \frac{\partial S_t}{\partial k_{t+1}} \frac{\partial k_{t+1}}{\partial k_t} = (1 + n) \frac{dk_{t+1}}{dk_t} - (1 - \delta_e), \]
where $\frac{\partial r_{t+1}}{\partial t} = \frac{\partial R_{t+1}}{\partial k_{t+1}}$ due to Eq. (13), rearranging terms taking into account Eq. (14)

$$\frac{dk_{t+1}}{dt} = \frac{1 - \delta_k + \frac{\partial S_t}{\partial W_t} \left( f_k - \frac{\partial R_t}{\partial k_t} - R_t \right)}{1 + n - \frac{\partial S_t}{\partial r_{t+1}} \frac{\partial R_{t+1}}{\partial k_{t+1}}}.$$  

The assumption made in Section 2.1 about $\frac{\partial S_t}{\partial r_{t+1}} \geq 0$ and Proposition 1 ensure that the denominator of this expression is positive. Thus, for the local stability of the equilibrium at a certain point $(c_t^0, c_{t+1}^0, k_t, g_t)$, we have to hold that:

$$D_t = n + \delta_k - \frac{\partial S_t}{\partial r_{t+1}} \frac{\partial R_{t+1}}{\partial k_{t+1}} - \frac{\partial S_t}{\partial W_t} k_t \frac{\partial R_t}{\partial k_t} + \frac{\partial S_t}{\partial W_t} > 0. \quad (20)$$

### 3 Steady state and welfare

In this section we are going to study how exogenous changes in $\tau$ and $g$ affect the main variables of the model in the long-run steady state, that is when both the private and public capital-labor ratio remain constant, $k_{t+1} = k_t = k$ and $g_{t+1} = g_t = g$. In order to ensure the local stability at the steady state equilibrium we assume that, exploiting (20),

$$D = n + \delta_k - \left( \frac{\partial S_t}{\partial r} - \frac{\partial S_t}{\partial W} k \right) \frac{\partial R_t}{\partial k} + \frac{\eta_g}{1 - \eta_g} f_k \frac{\partial S_t}{\partial W} > 0. \quad (21)$$

Therefore, in steady-state the equilibrium equation (19) can be written as

$$S(W, T^o, T^o, r) = (\delta_k + n)k. \quad (22)$$

Let us initially study the changes in the long-run steady state private capital-labor ratio that stem from an exogenous change in the weight of tax burden borne by each generation, holding the per-capita public investment constant. By taking the total derivative of (22) with respect to $\tau$, exploiting Eqs. (7) and (14),

$$\frac{dk}{d\tau} = \left( \frac{\delta_k + n}{D} \right) \left( \frac{\partial S}{\partial W} + \frac{1 + n}{1 + r} \left( 1 - \frac{\partial S}{\partial W} \right) \right).$$

This leads us to the following proposition:

**Proposition 2** Let $k(\tau, g)$ be the long-run equilibrium private capital-labor ratio corresponding to the parameters $\tau \in [0, 1]$ and $g > 0$. The amount of the private capital-labor ratio will be increased by increasing the share of tax burden bore by the older generation.
The proof is straightforward in view of (6) and (21). Thus, since \( \frac{dk}{d\tau} > 0 \), it follows immediately that \( \frac{dR}{d\tau} = \frac{\partial R}{\partial k} \frac{dk}{d\tau} < 0 \) due to Proposition 1. This result is consistent with the fact that savings increase the heavier the tax burden imposed on the older generation, stated in Eq. (7). The above expression also shows that if we are to maximize the long-run steady state private capital-labor ratio we should impose the entire tax burden on the older generation.

On the other hand, regarding the effects on the utility of an individual living in the long-run steady state of a change in the weight of tax burden borne by each generation, holding the per-capita public investment constant. From (2), let us define the indirect utility function

\[
V(W, r, T^y, T^o) = v[y(W, r, T^y, T^o)] + \beta v[c'(W, r, T^y, T^o)],
\]

Taking its total derivative with respect to \( \tau \) exploiting the first order condition for the individual’s equilibrium (4), we obtain

\[
\frac{dV}{d\tau} = \frac{\partial v}{\partial c^y} \left[ \left( \frac{\partial c^y}{\partial W} + \frac{1}{1 + r} \frac{\partial c^o}{\partial W} \right) \frac{dW}{d\tau} + \left( \frac{\partial c^y}{\partial r} + \frac{1}{1 + r} \frac{\partial c^o}{\partial r} \right) \frac{dR}{d\tau} + ... \right.
\]

\[
\left. ... + \left( \frac{\partial c^y}{\partial T^y} + \frac{1}{1 + r} \frac{\partial c^o}{\partial T^y} \right) \frac{dT^y}{d\tau} + \left( \frac{\partial c^y}{\partial T^o} + \frac{1}{1 + r} \frac{\partial c^o}{\partial T^o} \right) \frac{dT^o}{d\tau} \right].
\]

But, according to (3), \( \frac{\partial c^y}{\partial W} + \frac{1}{1 + r} \frac{\partial c^o}{\partial W} = 1, \frac{\partial c^y}{\partial r} + \frac{1}{1 + r} \frac{\partial c^o}{\partial r} = \frac{S}{1 + r}, \frac{\partial c^y}{\partial T^y} + \frac{1}{1 + r} \frac{\partial c^o}{\partial T^y} = -1 \) and \( \frac{\partial c^o}{\partial T^o} + \frac{1}{1 + r} \frac{\partial c^o}{\partial T^o} = -\frac{1}{1 + r} \). Therefore:

\[
\frac{dV}{d\tau} = \frac{\partial v}{\partial c^y} \left[ \frac{dW}{d\tau} + \frac{S}{1 + r} \frac{dR}{d\tau} - \frac{dT^y}{d\tau} - \frac{1}{1 + r} \frac{dT^o}{d\tau} \right].
\]

Finally, taking into account (14), (16), (18) and (22) we can write

\[
\frac{dV}{d\tau} = \frac{\partial v}{\partial c^y} \left[ -\frac{\eta_k}{1 - \eta_g} f_k \frac{dk}{d\tau} + \left( \frac{\delta_k + n}{1 + r} - 1 \right) k \frac{dR}{d\tau} + \left( \frac{\delta_g + n}{1 + r} \right) (r - n) g \right].
\]

It is easy to see from the above that, since \( \frac{dk}{d\tau} > 0 \) and \( \frac{dR}{d\tau} < 0 \), the sign of \( \frac{dV}{d\tau} \) depends on the value of the interest rate in equilibrium \( r \), regarding the value of the exogenous population growth rate \( n \), and the depreciation rate of private capital \( \delta_k \) in such a way that the following proposition can be made:

**Proposition 3** Let \( V(\tau, g) \) be the long-run level of utility corresponding to the parameters \( \tau \in [0, 1] \) and \( g > 0 \). The utility of an individual living in the long-run will be decreased by imposing a heavier tax burden on the older generation whenever \( r \leq n - (1 - \delta_k) \).

The proof is straightforward in view of Proposition 2 and the fact that \( r \leq n - (1 - \delta_k) \) is equivalent to \( \frac{\delta_k + n}{1 + r} \geq 1 \). In turn, provided that Proposition 3 tells us that is such a case \( \frac{dV}{d\tau} < 0 \), the entire tax burden should be imposed on
the younger generation if we are to maximize the utility of an individual living in the long-run steady state while holding \( g \) constant. On the other hand, if \( r \) does not fulfill such a condition, the sign of \( \frac{\partial V}{\partial r} \) becomes ambiguous.

Analogously, let us consider the changes in the long-run steady state of the private capital-labor ratio that stem from an exogenous change in the public investment policy. Taking the total derivative of (22) with respect to \( g \), we find that

\[
\frac{dk}{dg} = \frac{1}{D} \left( f_g \frac{\partial S}{\partial W} + \left( \frac{\partial S}{\partial r} - \frac{\partial S}{\partial W} k \right) \frac{\partial R}{\partial g} + \frac{\partial S}{\partial T^y} \frac{dT^y}{dg} + \frac{\partial S}{\partial T^o} \frac{dT^o}{dg} \right).
\]

Taking into account (7) and (14) the above expression can be written as

\[
\frac{dk}{dg} = \frac{1}{D} \left( f_g - (\delta_g + n) \left[ 1 + \tau \left( \frac{n - r}{1 + r} \right) \right] \right) \frac{\partial S}{\partial W} + \ldots \quad (24)
\]

\[
\ldots + \left( \frac{\partial S}{\partial r} - \frac{\partial S}{\partial W} k \right) \frac{\partial R}{\partial g} + \tau (\delta_g + n) \left( \frac{1 + n}{1 + r} \right).
\]

The variation in the steady state private capital-labor ratio due to changes in the public investment per-capita policy is here split into three parts. The ambiguity (or non-ambiguity) of its sign depends on whether taxes are (or are not) borne by only one generation or both, the net return on both public and private capital stocks, and the sign of \( \frac{\partial S}{\partial r} - \frac{\partial S}{\partial W} k \), provided that \( \frac{\partial S}{\partial r} > 0 \), \( \frac{\partial S}{\partial W} > 0 \). The sign of this expression is also related to the sign of (21) and, thus, to the local stability of the steady-state equilibrium. In what follows, without loss of generality, we are going to assume that \( \frac{\partial S}{\partial r} - \frac{\partial S}{\partial W} k > 0 \) Therefore, the following Proposition state conditions upon (24) has a clear-cut sign.

**Proposition 4** Let \( k(\tau, g) \) be the long-run equilibrium private capital-labor ratio corresponding to the parameters \( \tau \in [0,1] \) and \( g > 0 \). The amount of the private capital-labor ratio will be increased by increasing the amount of the public capital-labor ratio whenever:

(i) the older generation bears the entire tax burden (\( \tau = 1 \))

(ii) \( f_g - (\delta_g + n) \left[ 1 + \tau \left( \frac{n - r}{1 + r} \right) \right] \geq 0 \), when \( 0 \leq \tau < 1 \).

**Proof:** When \( \tau = 1 \) equation (24) becomes

\[
\frac{dk}{dg} = \frac{1}{D} \left[ f_g \frac{\partial S}{\partial W} + (\delta_g + n) \left( \frac{1 + n}{1 + r} \right) \left( 1 - \frac{\partial S}{\partial W} \right) + \left( \frac{\partial S}{\partial r} - \frac{\partial S}{\partial W} k \right) \frac{\partial R}{\partial g} \right],
\]

which is positive according to the former assumptions and propositions. Regarding the case \( 0 \leq \tau < 1 \), \( f_g - (\delta_g + n) \left[ 1 + \tau \left( \frac{n - r}{1 + r} \right) \right] \geq 0 \) becomes a more than sufficient condition for (24) to be positive.

The explanation of this result is that when public capital stock is financed only by taxes on the older generation, an increase in such taxes would increase...
savings, because the younger generation expects to pay more taxes in the future, with the direct consequence of an increase in private capital investment. On the other hand, when the tax burden is shared by both generations, the net return of public capital affects savings directly by means of wages and taxes when young. This means that in such a case a minimum net return of public capital is required to increase the private capital-labor ratio by increasing public investment per capita. For instance, in the extreme case in which \( \tau = 0 \) the necessary condition of part (ii) of Proposition 4 becomes \( f_g - (\delta_g + n) \geq 0 \), which means that in the long-run steady state the allocation of the public capital-labor ratio has to be dynamically efficient.\(^3\)

On the other hand, since \( \frac{dR}{dg} = \frac{\partial R}{\partial g} + \frac{\partial R}{\partial k} \frac{dk}{dg} \), it follows immediately, due to Propositions 1 and 4, that changes in public investment per capita policy on the return of private capital in equilibrium are ambiguous in general. Finally, regarding the effects of a change in the public investment policy on the utility of an individual living in the long-run steady state, let us assess the total derivative of (23) with respect to \( g \), exploiting the first order condition for the individual’s equilibrium (4),

\[
\frac{dV}{dg} = \frac{\partial v}{\partial c^y} \left[ \left( \frac{\partial c^y}{\partial W} + \frac{1}{1 + r} \frac{\partial c^o}{\partial W} \right) \frac{dW}{dg} + \left( \frac{\partial c^y}{\partial r} + \frac{1}{1 + r} \frac{\partial c^o}{\partial r} \right) \frac{dR}{dg} + \ldots + \left( \frac{\partial c^y}{\partial T^y} + \frac{1}{1 + r} \frac{\partial c^o}{\partial T^y} \right) \frac{dT^y}{dg} + \left( \frac{\partial c^y}{\partial T^o} + \frac{1}{1 + r} \frac{\partial c^o}{\partial T^o} \right) \frac{dT^o}{dg} \right].
\]

But, according to Eq. (3), \( \frac{\partial c^y}{\partial W} + \frac{1}{1 + r} \frac{\partial c^o}{\partial W} = 1, \frac{\partial c^y}{\partial c^o} + \frac{1}{1 + r} \frac{\partial c^o}{\partial c^o} = \frac{S}{1 + r}, \frac{\partial c^y}{\partial T^y} + \frac{1}{1 + r} \frac{\partial c^o}{\partial T^y} = -1 \) and \( \frac{\partial c^y}{\partial T^o} + \frac{1}{1 + r} \frac{\partial c^o}{\partial T^o} = -1 \).

\[
\frac{dV}{dg} = \frac{\partial v}{\partial c^y} \left[ \frac{dW}{dg} + \frac{S}{1 + r} \frac{dR}{dg} - \frac{1}{1 + r} \frac{dT^o}{dg} \right].
\]

\(^3\)Analogously to the case of two factors, for the optimal golden-age path we have to compute the path which maximizes the lifetime utility of the representative household subject to the economy-wide steady-state resource constraint:

\[
\max v(c^y) + \beta v(c^o)
\]

s.t. \( c^y + \frac{1}{1 + n} c^o = f(k, g) - (n + \delta_k)k - (n + \delta_g)g \).

The first-order conditions for the optimal golden-age path consist of the steady-state resource, the so-called biological-interest-rate consumption golden rule;

\[
\beta \frac{\partial v}{\partial c^o} \frac{\partial c^o}{\partial c^y} = \frac{1}{1 + n},
\]

and the production golden rule which defines the dynamic efficiency (Samuelson, 1968) for both private and public capital-labor ratio,

\[
f_k = n + \delta_k, \quad f_g = n + \delta_g.
\]
On the other hand, taking into account Eq. (14), \[ \frac{dW}{dg} = f_g - \frac{\eta_g}{1 - \eta_g} f_k \frac{df_k}{dg} - \frac{dR}{dg} k, \]
and (18) we have
\[
\frac{dV}{dg} = \frac{\partial v}{\partial c} y \cdot f_g - \left( \delta_g + n \right) \frac{\eta_g}{1 - \eta_g} f_k \frac{dk}{dg} + \left( \frac{\delta_k + n}{1 + r} - 1 \right) k \frac{\partial R}{\partial g}.
\]

As we can see, the sign of (25) is ambiguous in general as long as the signs of its second and third part are related to the sign of \( \frac{\delta_k + n}{1 + r} \). On the one hand, when \( \frac{\delta_k + n}{1 + r} \geq 1 \) the third part of (25) is non-negative but the second is negative; this fact, together with the positiveness or negativeness of the first part, makes the sign of (25) ambiguous in such a case. On the other hand, when \( \frac{\delta_k + n}{1 + r} \leq 1 \), by taking into account Propositions 1 and 4, one can set conditions to hold a clear-cut result about the sign of (25). For instance, the third part of (25) depends on \( \frac{\partial R}{\partial g} \), which is positive according to Proposition 1, and the second part depends on \( \frac{dk}{dg} \), which is also positive when \( \tau = 1 \) according to Proposition 4. This allow us to state the following Proposition:

**Proposition 5** Let \( V(\tau, g) \) be the long-run level of utility corresponding to the parameters \( \tau \in [0, 1] \) and \( g > 0 \). The utility of an individual living in the long-run will be decreased by increasing the amount of the public capital-labor ratio whenever the older generation bears the entire tax burden \( (\tau = 1) \) and:

\[
1 - \frac{(1 - \eta_g)\eta_g}{(1 - \eta_k)\eta_k - (1 - \alpha)\eta_k^k} \leq \frac{\delta_k + n}{1 + r} \leq 1,
\]

\[
\frac{f_g}{1 + n} \leq \frac{\delta_g + n}{1 + r}.
\]

Proof: When \( \tau = 1 \), the first part of (25) is non-positive if \( \frac{f_g}{1 + n} \leq \frac{\delta_k + n}{1 + r} \). On the other hand, the third part is non-positive if \( \frac{\delta_k + n}{1 + r} \leq 1 \) since \( \frac{\partial R}{\partial g} > 0 \); and, as long as \( \frac{dk}{dg} > 0 \), the second part of (25) is non-positive if \( \left( \frac{\delta_k + n}{1 + r} - 1 \right) k \frac{\partial R}{\partial k} \leq \frac{\eta_g}{1 - \eta_g} f_k \), substituting the value of \( \frac{\partial R}{\partial k} \) assessed in the appendix the result is held.

Note that assuming that \( \frac{(1 - \eta_g)\eta_g}{(1 - \eta_k)\eta_k - (1 - \alpha)\eta_k^k} < 1 \) (assumption which is fulfilled for a nested CES-type aggregated production function when both private and public capital stock are not substitutes), the two final conditions of Proposition 5 can be reduced to the following single condition on the interest rate in equilibrium:

\[
\delta_k + n - 1 \leq r \leq \min \left\{ \Psi \left( \delta_k + n \right), \frac{T^\alpha}{\eta_g f_k} \right\} - 1,
\]
where $\Psi = \frac{(1-\eta_k)(1-\alpha)\eta_k^L}{(1-\eta_k)(1-\alpha)(\eta_k^L - (1-\eta_g)\eta_g)} > 1$. It goes without saying that in such a case it is also necessary that $\delta_k + n < \frac{N}{\eta_g}$ in order to ensure a non-empty interval for $r$. Summarizing, we have stated some reasonable particular conditions on the interest rate in equilibrium for which the utility of the representative individual living in the long-run steady state may decline with rises in the public investment per-capita policy. On the other words, such a policy does not improve long-run welfare in general.

4 Final comments

In this paper we have used an overlapping generation model to analyze some long-run steady state effects prompted by a free-access public capital when it enters a constant-returns-to-scale aggregated production function. The two contributions in the formalization of our model are, on the one hand, the characterization of the congestion problem arising from the common property by means of Feehan and Batina’s (2007) price rule (where public capital returns are dissipated among the rest of the private factors), and, on the other, the funding of public investment by sharing the tax burden between younger and older generations. At that point, given the positive effect that a tax rate on the older generation has on savings, we had to assume that the substitution effect dominates the income one, to ensure the stability of the model. As a consequence of that we found results which concern with both changes in the share of tax burden, by shifting it from the younger to the older generation, holding public investment constant, and changes in the constant public investment policy, by increasing the public capital-labor ratio. In the first case we found that a shift of the tax burden from the younger to the older generation increases the amount of the long-run steady state private capital-labor ratio. In addition, such a change may decrease the utility of an individual living in the long-run steady state whenever the equilibrium interest rate is low enough. These effects are doubtlessly a consequence of the above-commented positive dependence of savings with respect to taxes when old. Moreover, regarding the case of changes in the constant public investment policy, the results are much more ambiguous since private factor prices depend on public capital returns in an intricate way. Nevertheless, some results have been reached by setting conditions on the share of tax burden borne by each generation, the net return of public capital and the interest rate in equilibrium. In particular we found that whenever the older generation bears the entire tax burden an increase in the public capital-labor ratio leads to an increase in the long-run steady state private capital-labor ratio. On the other hand, when the entire tax burden is borne by the younger generation, this increase depends on the fact that the long run steady state public capital-labor ratio will be dynamically efficient. Finally, we end by concluding that, in general, an increase in the constant public investment per-capita policy may lead to a decline in the utility of an individual living in the long run. This conclusion depends on some conditions, such as the fact that the entire tax bur-
den has to be borne by the older generation, and the equilibrium interest rate has to be bounded in a certain interval which depends on exogenous parameters and the product elasticities of both private and public capital stocks.

Appendix

Proof of Proposition 1.

For \( \frac{\partial R}{\partial g} > 0 \), taking the partial derivative of (16) with respect to \( g \) (subindex \( t \) is dropped),

\[
\frac{\partial R}{\partial g} = \frac{(f_{kg} + f_k f_g)(f - g f_g) - f_k f (f_g - f_g g)}{(f - g f_g)^2},
\]

rearranging terms and taking the notation in terms of the elasticities (9)

\[
\frac{\partial R}{\partial g} = \frac{f_g}{k_t(1 - \eta_g)^2} \left[ \eta^f_k (1 - \eta_g) + \eta_k \eta^f_g + (1 - \eta_g) \eta_k \right],
\]

taking into account (10) it can be written as

\[
\frac{\partial R}{\partial g} = \frac{f_g}{k_t(1 - \eta_g)^2} \left[ (1 - \alpha + \eta_k) + \eta_k \eta^f_g + (1 - \alpha + \eta_k) \eta_k \right],
\]

rearranging terms

\[
\frac{\partial R}{\partial g} = \frac{f_g}{k_t(1 - \eta_g)} \left[ \eta^f_k (1 - \alpha) + \eta_k \eta^f_g + \eta^f_k (1 - \alpha) + \eta_k \eta^f_g \right],
\]

exploiting (12) and clearing

\[
\frac{\partial R}{\partial g} = \frac{f_g}{k_t(1 - \eta_g)^2} \left[ (1 - \alpha) \eta_k^f + \eta_k^f \right] > 0.
\]

For \( \frac{\partial R}{\partial k} < 0 \), taking the partial derivative of (16) with respect to \( k \) (subindex \( t \) is dropped),

\[
\frac{\partial R}{\partial k} = \frac{(f_{kk} f + f^2_k)(f - g f_g) - f_k f (f_k - g f_{kg})}{(f - g f_g)^2},
\]

rearranging terms and taking the notation in terms of the elasticities (9)

\[
\frac{\partial R}{\partial k} = \frac{f_k}{k_t(1 - \eta_g)^2} \left[ \eta^f_k (1 - \eta_g) + \eta_k \eta^f_g - \eta_k \eta_g \right],
\]

taking into account (10) it can be written as

\[
\frac{\partial R}{\partial k} = \frac{f_k}{k_t(1 - \eta_g)^2} \left[ \eta^f_k (1 - \alpha) + \eta_k \eta^f_g - \eta_k \eta_g \right],
\]

exploiting (11), and rearranging terms

\[
\frac{\partial R}{\partial k} = \frac{f_k}{k_t(1 - \eta_g)^2} \left[ \eta^f_k (1 - \alpha) - (1 - \alpha + \eta_g) \eta_k \right],
\]
finally, taking into account (10) it can be written as:

\[
\frac{\partial R_t}{\partial k_t} = \frac{f_k}{k_t(1 - \eta_k)^2} \left[ (1 - \alpha)\eta_k^f - (1 - \eta_k)\eta_k \right] < 0. \quad \blacksquare
\]

References


