GENERIC DETERMINACY OF NASH EQUILIBRIUM IN NETWORK-FORMATION GAMES

CARLOS PIMENTA†

Abstract. This paper proves the Generic determinacy of Nash equilibrium in network-formation games: for a generic assignment of utilities to networks, the set of probability distributions on networks induced by Nash equilibria is finite.

1. Introduction

A basic desideratum when applying noncooperative game theory is to have a finite set of probability distributions on outcomes derived from equilibria.1 When utilities are defined over the relevant outcome space, it is well known that this is generically the case when we can assign a different outcome to each pure strategy profile (Harsanyi, 1973), or to each ending node of an extensive-form game (Kreps and Wilson, 1982).2

A game-form endows players with finite strategy sets and specifies which is the outcome that arises from each pure strategy profile.3 It could identify, for instance, two ending nodes of an extensive-form game with the same outcome. Govindan and McLennan (2001) give an example of a game-form such that, in an open set of utilities over outcomes, produces infinitely many equilibrium distributions on outcomes. In view of such a negative result, we have to turn to specific classes of games to seek for positive results regarding the generic determinacy of the Nash equilibrium concept. For some examples, see Park (1997) for sender-receiver games, and De Sinopoli (2001) and De Sinopoli and Iannantuoni (2005) for voting games.

1School of Economics, The University of New South Wales, Sydney 2052, Australia. E-mail address: c.pimenta@unsw.edu.au.
Date: October 12, 2008.
JEL Classification. C62, C72, D85.
Key words and phrases. Networks, Generic Finiteness, Nash Equilibrium.
* This work is a revised version of the third chapter of my PhD dissertation for the Universidad Carlos III de Madrid. I am deeply grateful to my supervisors Luis Corchén and Francesco De Sinopoli for their continuous encouragement and helpful comments. I am also thankful to the advisory editor and an anonymous referee for numerous suggestions that significantly improved the exposition of the paper and originated Sections 5.3 and 5.4. I acknowledge financial support from the Australian School of Business SRG grant scheme and from the Centre for Applied Economic Research (CAER). The usual disclaimer applies.

1By outcomes we mean the set of physical or economic outcomes of the game (i.e. the set of different economic alternatives that could be found after the game is played) and not the set of probability distribution induced by equilibria. We will refer to the latter concept as the set of equilibrium distributions.

2Harsanyi (1973) actually proves that the set of Nash equilibria is finite for a generic assignment of payoffs to pure strategy profiles.

3More generally, it specifies a probability distribution on the set of outcomes. Game-forms are formally defined in Section 2.2.
This paper studies the generic determinacy of the Nash equilibrium concept when individual payoffs depend on the network connecting them. The network literature has been fruitful to describe social and economic interaction. See for instance Jackson and Wolinsky (1996), Kranton and Minehart (2001), Jackson and Watts (2002), or Calvo-Armengol (2004). It is, therefore, important to have theories that explain how such networks form. Different network-formation procedures have been proposed. For a comprehensive survey of those theories the reader is referred to Jackson (2005).

One of the most used network-formation games was suggested by Myerson (1991) and it can be described as follows. Each player simultaneously proposes a list of players with whom she wants to form a link and a direct link between two players is formed if and only if both players agree on that. Calvo-Armengol and Ilkilic (2007) study this game to provide a connection between pairwise-stability, a network equilibrium concept defined in Jackson and Wolinsky (1996), and proper equilibrium, a noncooperative equilibrium refinement due to Myerson (1978).

This game is simple and intuitive, however, since it takes two players to form a link, a coordination problem arises which makes the game exhibit multiplicity of equilibria. This fact constitutes a common concern in network theory. Nevertheless, we can prove that even though a network-formation game may have a large number of equilibria, generically, the set of probability distributions on networks that they induce is finite.

The network-formation game is formally presented in the next section. Section 3 discusses an example. Section 4 contains the main result and its proof. To conclude, Section 5 discusses some remarks, extensions of the result to other network-formation games and a related result for the extensive-form game of network-formation introduced by Aumann and Myerson (1989).

2. Preliminaries

Given a finite set $A$, denote as $\mathcal{P}(A)$ the set of all subsets of $A$, and as $\Delta(A)$ the set of probability distributions on $A$.

2.1. Networks. Given a set of players $N = \{1, \ldots, n\}$, a network $g$ is a collection of direct links. A direct link in the network $g$ between two different players $i$ and $j$ is denoted by $ij \in g$. For the time being we focus on undirected networks. In an undirected network $ij \in g$ is equivalent to $ji \in g$.\footnote{In a directed network, if $i$ and $j$ are two different agents, the link $ij$ is different from the link $ji$. This two links can be regarded as different if, for instance, they explain which is the direction of information, or which is the player who is sponsoring the link.}

The set of $i$'s direct links in $g$ is $L_i(g) = \{jk \in g : j = i \text{ or } k = i\}$. The complete network $g^N$ is such that $L_i(g^N) = \{ij : j \neq i\}$, for all $i \in N$. In $g^N$ player $i$ is directly linked to every other player. The set of all undirected networks on $N$ is $\mathcal{G} = \mathcal{P}(g^N)$.

Each player $i$ can be directly linked with $n - 1$ other players. The number of links in the complete network $g^N$ is $n(n - 1)/2$, dividing by 2 not to count links twice. Since $\mathcal{G}$ is the power set of $g^N$, it has $2^{n(n-1)/2}$ elements.

\footnote{In graph theory, the kind of networks defined here are called simple undirected graphs.}
2.2. Game-Forms. A game-form is given by a set of players \( N = \{1, \ldots, n\} \), nonempty finite sets of pure strategies \( S_1, \ldots, S_n \), a finite set of outcomes \( \Omega \), a function \( \theta : S \to \Delta(\Omega) \), and utilities defined over the outcome space \( \Omega \), that is, \( u_1, \ldots, u_n : \Omega \to \mathbb{R} \). Once we fix \( N, S_1, \ldots, S_n, \Omega, \) and \( \theta \), a game-form is given by a point in \((\mathbb{R}^\Omega)^N\).

Utility functions \( u_1, \ldots, u_n \) over \( \Omega \) induce utility functions \( v_1, \ldots, v_n \) over \( S = S_1 \times \cdots \times S_n \) according to \( u_1 \circ \theta, \ldots, u_n \circ \theta \). Hence, every game-form has associated its finite normal form game.

2.3. The Network-Formation Game. The set of players is \( N \). All players in \( N \) simultaneously announce the set of direct links they wish to form. Formally, the set of player \( i \)'s pure strategies is \( S_i = \mathcal{P}(N \setminus \{i\}) \). Therefore, a strategy \( s_i \in S_i \) is a subset of \( N \setminus \{i\} \) and is interpreted as the set of players other than \( i \) with whom player \( i \) wishes to form a link. Mutual consent is needed to create a direct link, i.e., if \( s \) is played, \( ij \) is created if and only if \( j \in s_i \) and \( i \in s_j \).

We can adapt the previous general description of game-forms to the present context in order to specify the game-form that structures the network-formation game. Let the set of players and the collection of pure strategy sets be as above. The set of outcomes is the set of undirected networks, i.e., \( \Omega = \mathcal{G} \). The function \( \theta \) is a deterministic outcome function, formally, \( \theta : S \to \mathcal{G} \). Given a pure strategy profile, \( \theta \) specifies which network is formed respecting the rule of mutual consent to create direct links. Utilities are functions \( u_1, \ldots, u_n : \mathcal{G} \to \mathbb{R} \). Once the set of players \( N \) is given, the pure strategy sets are automatically created and the network-formation game is defined by a point in \((\mathbb{R}^\mathcal{G})^N\).

If players other than \( i \) play according to \( s_{-i} \in S_{-i} \), \( 6 \) the utility to player \( i \) from playing strategy \( s_i \) is equal to \( u_i(s_{-i}, s_i) = u_i(\theta(s_{-i}, s_i)) \).

Let \( \Sigma_i = \Delta(S_i) \) be the set of mixed strategies of player \( i \). Furthermore, let \( \Sigma = \Sigma_1 \times \cdots \times \Sigma_n \). While a pure strategy profile \( s \) results in the network \( \theta(s) \) with certainty, a mixed strategy profile \( \sigma \) generates a probability distribution on \( \mathcal{G} \), where the probability that \( g \in \mathcal{G} \) forms equals

\[
P(g | \sigma) = \sum_{s \in \theta^{-1}(g)} \left( \prod_{i \in N} \sigma_i(s_i) \right).
\]

If players other than \( i \) play according to \( \sigma_{-i} \in \Sigma_{-i} \), \( 7 \) the utility to player \( i \) from playing the mixed strategy \( \sigma_i \) is equal to \( V_i(\sigma_{-i}, \sigma_i) = \sum_{g \in \mathcal{G}} P(g | (\sigma_{-i}, \sigma_i)) u_i(g) \).

**Definition 1** (Nash Equilibrium). The strategy profile \( \sigma \in \Sigma \) is a Nash equilibrium of the network-formation game if \( V_i(\sigma_{-i}, \sigma_i) \geq V_i(\sigma_{-i}, \sigma_i') \) for all \( \sigma_i' \in \Sigma_i \), and for all \( i \in N \).

2.4. Generic Finiteness of Equilibrium Distributions in game-forms. Let us first give the definition of a generic set.

**Definition 2.** For any \( m \geq 0 \), we say that \( G \subset \mathbb{R}^m \) is a generic set, or generic, if \( \mathbb{R}^m \setminus \text{int}(G) \) has Lebesgue measure 0.

---

6\( S_{-i} = \prod_{j \neq i} S_j \).

7\( \Sigma_{-i} = \prod_{j \neq i} \Sigma_j \).
Govindan and McLennan (2001) give an example of a game-form that, in an open set of utilities over outcomes, produces infinitely many equilibrium distributions on the outcome space. Nevertheless, they also provide a number of positive results. Their Theorem 5.3 generalizes Harsanyi’s result to game-forms where at every strategy vector the set of probability distribution on outcomes that every player can induce by changing her strategy has the same dimension as her mixed strategy set. As they notice, for this condition to be true “the game-form cannot identify two pure strategy vectors with the same outcome if they specify different pure strategies for only one agent”. Consequently, their result cannot be applied to the network-formation game described in 2.3 (see Table 1 below for an example). We need a modification of their result. Consider the general specification of game-forms given in Section 2.2:

**Proposition.** If $\theta$ is such that at all completely mixed strategy tuples and for each agent $i$ the set of distributions on $\Omega$ that agent $i$ can induce by changing her strategy is $(|S_i|-1)$-dimensional, then for generic utilities there are finitely many completely mixed Nash equilibria. Moreover, the set of utilities that induce a continuum of completely mixed Nash equilibria is a lower-dimensional semi-algebraic set.

The proof is offered in the Appendix.

3. An Example

Consider a 3-person network-formation game. The corresponding game-form is depicted in Table 1. Player 1 is the row player, player 2 the column player, and player 3 the matrix player. The symbol $g^0$ denotes the empty network, $g^N$ denotes the complete network, $g_{ij}$ denotes the network that only contains link $ij$, and $g^i$ denotes the network where player $i$ is connected to every other player and such that there are no further links.

<table>
<thead>
<tr>
<th></th>
<th>$\emptyset$</th>
<th>${1}$</th>
<th>${3}$</th>
<th>${1,3}$</th>
<th>$\emptyset$</th>
<th>${1}$</th>
<th>${3}$</th>
<th>${1,3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
</tr>
<tr>
<td>${2}$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
</tr>
<tr>
<td>${3}$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
</tr>
<tr>
<td>${2,3}$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
<td>$g^0$</td>
</tr>
</tbody>
</table>

Table 1. The game-form of a network-formation game with three players.

Suppose that the utility function of player $i = 1, 2$ is $u_i(g) = |L_i(g)|$, i.e. player $i = 1, 2$ derives an utility from network $g$ equal to the number of direct links that
she maintains in $g$. Suppose also that player 3 has the same utility as players 1 and 2, except that she derives an utility equal to 2 from network $g^2$. Specifically,

$$u_i(g^0) = 0 \text{ for all } i,$$

$$u_i(g^{jk}) = \begin{cases} 1 \text{ if } i = k \text{ or } i = j \\ 0 \text{ otherwise,} \end{cases}$$

$$u_i(g^j) = \begin{cases} 2 \text{ if } i = j \\ 2 \text{ if } i = 3 \text{ and } j = 2 \\ 1 \text{ otherwise,} \end{cases}$$

$$g_i(g^N) = 2 \text{ for all } i.$$

Table 2 displays the set of Nash equilibria of this game. The subset of Nash equilibria of line (i) supports the empty network, the subsets of line (ii) support, respectively, networks $g^{12}$, $g^{13}$ and $g^{23}$, the subsets of line (iii) support, respectively, networks $g^1$, $g^2$ and $g^3$.

(i) \[ \text{NE} = \left\{ (\emptyset, \emptyset, \emptyset) \right\} \cup \]

(ii) \[ \left\{ ((2), (1), \emptyset) \right\} \cup \left\{ ((3), \emptyset, (1)) \right\} \cup \left\{ (\emptyset, (3), \{2\}) \right\} \cup \]

(iii) \[ \left\{ ((2,3), (1), (1)) \right\} \cup \left\{ ((2), (1,3), \{2\}) \right\} \cup \left\{ ((3), (1,2)) \right\} \cup \]

(iv) \[ \left\{ ((2,3), (1,3), \lambda\{2\} + (1 - \lambda)\{1,2\}) : \lambda \in [0,1) \right\}. \]

Table 2. Set of Nash equilibria of the 3-person network-formation game discussed in Section 3.

The subset of equilibria of line (iv) induces a continuum of probability distribution over the set of networks that give probability $\lambda$ to network $g^2$ and probability $(1 - \lambda)$ to the complete network $g^N$ for $\lambda \in [0,1]$.

Now perturb independently the utility that each player obtains from each network. The subsets of strategy profiles of lines (i) through (iii) are still equilibrium strategy profiles. In addition, there are two possibilities:

- Player 3 ranks the complete network $g^N$ over network $g^2$. In this case the set of Nash equilibria is composed of lines (i) through (iii) united to \[ \left\{ ((2,3), (1,3), (1,2)) \right\}, \]

which supports the complete network.

- Player 3 ranks network $g^2$ over the complete network $g^N$. Then, no Nash equilibrium gives positive probability to the complete network. The set of Nash equilibria is composed of lines (i) through (iii) united to \[ \left\{ (\lambda\{2\} + (1 - \lambda)\{2,3\}, (1,3), \{2\}) : \lambda \in [0,1) \right\}, \]

which supports network $g^2$.

In either case, there is a finite number of probability distributions on networks induced by equilibria.
4. The Result

The discussion of the case where player 3 ranks network \( g^2 \) over the complete network illustrates that we cannot hope for generic finiteness of equilibrium points in network-formation games. A direct application of the Proposition tells us that there is a finite set of completely mixed equilibrium points, but it leaves open the possibility that there be a continuum of equilibria in which some pure strategy of some player remains unused.

The following proof truncates the network-formation game into smaller games in which some strategies are not available, and in which every link proposal meets a positive response if the counterpart of the link plays a completely mixed strategy. Each of these smaller games has a finite number of completely mixed equilibria. The desired result comes from the fact that every equilibrium of a truncated game that does not belong to the previous family induces the same probability distribution on networks as some equilibrium of some game that does belong to such a family.

**Theorem.** For generic \( u \in (\mathbb{R}^G)^N \) the set of probability distributions on networks induced by Nash equilibria of the network-formation game is finite. Moreover, the set of payoffs for which this result does not hold is a lower-dimensional semi-algebraic set.

**Proof.** Given a network-formation game, there is a finite number of different normal form games obtained by assigning to each player \( i \) an element of \( \mathcal{P}(S_i) \) as her strategy set.

Let \( T = T_1 \times \cdots \times T_n \), where \( T_i \subseteq S_i \). The normal form game \( \Gamma_T \) is defined by the set of players \( N \), the collection of strategy sets \( \{T_i\}_{i \in N} \), and the collection of utility functions \( \{v_i^T\}_{i \in N} \), where \( v_i^T \) is the restriction of \( v_i \) to \( T \). Furthermore, let \( \mathcal{G}_T = \theta(T) \).

It is enough to prove that for a generic assignment of payoffs to networks, completely mixed Nash equilibria of each of those games induce a finite set of probability distributions on \( \mathcal{G}_T \). Notice that every equilibrium of any game can be obtained as a completely mixed equilibrium of the modified game obtained by eliminating unused strategies.

Consider the game \( \Gamma_T \). Given player \( i \), the set \( D_i = \{ j \in N : i \notin t_j \text{ for all } t_j \in T_j \} \) is the set of players that do not have player \( i \) in any of their strategies. Let \( \hat{T}_i = \{t_i \setminus D_i : t_i \in T_i\} \) and let \( \hat{T} = \hat{T}_1 \times \cdots \times \hat{T}_n \). Notice that for each completely mixed Nash equilibrium of \( \Gamma_T \), there exists a completely mixed Nash equilibrium of \( \Gamma_{\hat{T}} \) that induces the same probability distribution on \( \mathcal{G}_{\hat{T}} \).

At every completely mixed strategy profile \( \sigma \) of \( \Gamma_{\hat{T}} \), every network in \( \mathcal{G}_{\hat{T}} \) receives positive probability. At the strategy profile \( (t_i, \sigma_{-i}) \), only those networks \( g \in \mathcal{G}_T \) such that \( \{ij : j \in t_i\} \subset g \) and \( \{ij : j \notin t_i\} \cap g = \emptyset \) are possible. Since for every player \( i \) each of her pure strategies means a different set of proposed links, we have that:

\[
\text{rank } \frac{\partial \mathbf{P}}{\partial \sigma_i}(\cdot | \sigma) = |\hat{T}_i| - 1.
\]

Therefore, at every completely mixed strategy profile of \( \Gamma_T \) the set of probability distributions on \( \mathcal{G}_T \) that player \( i \) can induce by varying her strategy is \((|\hat{T}_i| - 1)\)-dimensional. We can apply the Proposition to the game-form given by \( \hat{T} \) and \( \theta_{\hat{T}} \), the restriction of \( \theta \) to \( \hat{T} \). This implies that for generic utilities over \( \mathcal{G}_T \) there are...
finitely many completely mixed equilibria of $\Gamma_T$, which in turn implies that the set of probability distributions on $\mathcal{G}_T$ induced by completely mixed Nash equilibria of $\Gamma_T$ is generically finite.

Let $T \subseteq S$, we can write $(\mathbb{R}^\mathcal{G})^N = (\mathbb{R}^{\mathcal{G}_T})^N \times (\mathbb{R}^{\mathcal{G}\setminus\mathcal{G}_T})^N$. Let $K$ be a closed lower-dimensional semi-algebraic set in $(\mathbb{R}^{\mathcal{G}_T})^N$, i.e., the set of payoffs over $\mathcal{G}_T$ such that the set of completely mixed Nash equilibria of $\Gamma_T$ induces infinitely many probability distributions on $\mathcal{G}_T$, then the closed set $K \times (\mathbb{R}^{\mathcal{G}\setminus\mathcal{G}_T})^N$ is a lower-dimensional semi-algebraic set in $(\mathbb{R}^\mathcal{G})^N$. The same is true for any other $T' \subseteq S$. This concludes the proof. □

5. Remarks

5.1. Absence of Mutual Consent. Models of network-formation can be found in the literature that do not require common agreement between the parties to create a direct link, see for instance Bala and Goyal (2000). Thus, suppose that mutual consent is not needed to create a direct link. Let $N$ be the set of players, let $S_1, \ldots, S_n$ be the collection of pure strategy sets, where $S_i = \mathcal{P}(N \setminus \{i\})$ for all $i$ in $N$, and let $\mathcal{G}$ be the outcome space. In the model analyzed in Section 4, a link may not be created even if a player wants it to be created. In the current model, a link may be created even if a player does not want it to be created. In this modified network-formation game, generically, the set of equilibrium distributions on $\mathcal{G}$ is also finite. Notice that we can reinterpret pure strategies $s_i \in S_i$ as the set of players other than $i$ with whom player $i$ does not wish to form a link. The link $ij$ is not created only if player $i$ does not want to be linked with player $j$ and player $j$ does not want to be linked with player $i$. Define $\theta': S \to \mathcal{G}$ according to $\theta'(s) = g^N \setminus \theta(s)$, where $\theta$ is as defined in Section 2.3. Now, apply the proof of Section 4.

5.2. Directed Networks. Sometimes links $ij$ and $ji$ cannot be treated as equivalent for reasons coming from the nature of the phenomena being modeled. Directed networks respond to this necessity.

Denote the set of directed networks as $\mathcal{G}^d$. Suppose first that link formation does not need mutual consent. The strategy set of player $i$ is $S_i = \mathcal{P}(N \setminus \{i\})$. A strategy $s_i \in S_i$ is interpreted as the set of players other than $i$ with whom player $i$ wants to start an arrowhead link pointing at herself, i.e. the set of links that player $i$ wishes to receive.\(^{10}\)

Notice that each pure strategy profile leads to a different element in $\mathcal{G}^d$: each player has $2^{n-1}$ pure strategies, and there are $2^{n(n-1)}$ directed networks. Therefore, we are in the case of normal form payoffs where the generic finiteness of equilibria is guaranteed.

Suppose now that if a player $i$ wants to receive a link from player $j$, player $j$ needs to declare that she wants to send a link to player $i$ for it to be created. To accommodate for this case, let the strategy set of player $i$ be $S_i = S^{r}_i \times S^{s}_i = \mathcal{P}(N \setminus \{i\}) \times \mathcal{P}(N \setminus \{i\})$. A strategy $s_i \in S_i$ has two components, $s^{r}_i$ and $s^{s}_i$. We interpret $s^{r}_i$ as the set of players other than $i$ from whom player $i$ wishes to receive a link, and $s^{s}_i$ as the set of players other than $i$ to whom player $i$ wishes to send a link.

\(^{10}\)We can assume, for instance, that the arrowhead tells which is the direction of the flow of information.
link. Suppose that the pure strategy profile $s$ is played. The link $ij$ is created only if $j \in s_i^j$ and $i \in s_i^j$.

A similar proof to the one used in Section 4 establishes the generic determinacy of the Nash equilibrium concept under this setting. The key step that we must change is the following: Let $T = T_1 \times \cdots \times T_n$ where $T_i \subseteq S_i$ for all $i$ and consider the normal form game $\Gamma_T$. Let $D_i^m$ denote the set of players that in $\Gamma_T$ cannot send a link to player $i$. Let $D_i^m$ denote the set of players that in $\Gamma_T$ cannot accept a link sent by $i$. The new game $\Gamma_{T}$, where $\hat{T}_i = \{(t_i^1 \setminus D_i^m, t_i^2 \setminus D_i^m) : (t_i^1, t_i^2) \in T_i\}$, satisfies the hypothesis of the Proposition and, consequently, the desired result follows.

5.3. Links of Different Types. Here we consider network-formation games where players can establish more than one link between one another and links are of different types. Within this generalization, let us focus on undirected Networks.

Suppose that $M$ is a finite set of different types of links. In the associated network-formation game, player $i$’s strategy set is $S_i = \mathcal{P}(N \setminus \{i\})^M$. If the strategy profile $s$ is played, player $i$ and player $j$ establish a link of type $m \in M$ if $j \in s_i^m$ and $i \in s_j^m$, where $s_i^m$ and $s_j^m$ are the $m$th component of, respectively, $s_i$ and $s_j$.

A modification of the main proof along the same lines as the one discussed at the end of Section 5.2 accounts for the finiteness of equilibrium distributions on networks when links can be of several types. Given any $T \subseteq S$, for each player $i$ and for each type of link $m$, let $D_i^m$ denote set of players that in the game $\Gamma_T$ cannot propose a link of type $m$ to player $i$. We can apply the Proposition to $\Gamma_{T}$, where $\hat{T}_i^m = \{t_i^m \setminus D_i^m : t_i^m \in T_i\}$ for all $m \in M$ and for every player $i \in N$.

5.4. Restrictions in the Set of Possible Links. In many settings it is natural to restrict the set of possible links to some subset of the complete network.

When this is the case, players’ strategy sets of the associated network-formation game must be defined accordingly, thus, generating one of the truncated games considered in the proof in Section 4. Consequently, an analogous proof to the one offered there guarantees the generic finiteness of equilibrium distributions.

5.5. An Extensive-Form Game of Network-Formation. We have focused on normal form games of network-formation. However, there exists a prominent extensive game of network-formation due to Aumann and Myerson (1989). They proposed the first explicit formalization of network-formation as a game. It relies on an exogenously given order over possible links. Let $(i_1, j_1, \ldots, i_m, j_m)$ be such a ranking.

The game has $m$ stages. In the first stage players $i_1$ and $j_1$ move simultaneously to decide whether or not they form link $i_1j_1$. Each of them chooses an action from the set $\{yes, not\}$. The link $i_1j_1$ is established if and only if both players choose yes. Once the decision on link $i_1j_1$ is taken, every player gets informed about it, and the play of the game moves to the decision about link $i_2j_2$. The game evolves in the same fashion, and finishes with the stage where players $i_m$ and $j_m$ decide upon link $i_mj_m$.\footnote{If players get informed about which has been the terminal position in the simultaneous move game of every stage, the same argument offered below also goes through. Several other features can also be added to this basic model. For instance, two players can be called to reconsider their decision in case some set of links is formed, or two player may not be allowed to decide upon the link connecting them. At this respect, if players are forming an undirected network, $m$ can be different from $2^{n(n-1)/2}$.} The resulting network is formed by the set links $i_kj_k$ such that
both players \(i_k\) and \(j_k\) chose yes at stage \(k\). Although in the argument we work with undirected networks, the game can be applied to the formation of directed networks.

The argument that follows is a modification of the one used by Govindan and McLennan (2001) to prove that, for a given assignment of outcomes to ending nodes in an extensive game of perfect information, and for utilities such that no player is indifferent between two different outcomes, every Nash equilibrium induces a degenerate probability distribution in the set of outcomes. Such an argument is, in turn, a generalization of the one used by Kuhn (1953) to prove his “backwards induction” theorem that characterizes subgame perfect equilibria for games of perfect information.

Consider the generic set of utilities

\[
U_G = \left\{ u \in (\mathbb{R}^G)^N : u_i(g_1) \neq u_i(g_2) \text{ for all } i \in N \text{ and all } g_1, g_2 \in G \right\}.
\]

The claim is that if the utility vector is \(u \in U_G\), every Nash equilibrium induces a probability distribution on \(G\) that assigns probability one to some \(g \in G\).

Let \(S_i\) denote the set of pure strategies of player \(i\), where now a pure strategy is a function that assigns one element of \(\{\text{yes}, \text{not}\}\) to each information set of player \(i\). As usual, \(\Sigma_i = \Delta(S_i)\) and \(\Sigma = \Sigma_1 \times \cdots \times \Sigma_n\).

Let \(\sigma \in \Sigma\) be a Nash equilibrium for \(u \in U_G\). The appropriate modification of \(\sigma\), say \(\bar{\sigma}\), is a completely mixed Nash equilibrium of the extensive-form game obtained by eliminating all information sets and branches that occur with zero probability in case \(\sigma\) is played. In this reduced game, every information set has a well defined conditional probability over networks and, obviously, \(\bar{\sigma}\) induces the same probability distribution on \(G\) as \(\sigma\).

If there is a stage where a player randomizes between yes and not and the other player does chooses yes with positive probability, there must be a last such stage. But at this last stage, say \(i_hj_h\), such an agent, say \(i_h\), cannot be optimizing, since she is not indifferent between \(g \setminus \{i_hj_h\}\) and \(g \cup \{i_hj_h\}\) for any \(g \in G\).

We can adapt the previous argument to the case where mutual consent is not needed to create a link. Let \((i_1j_1, \ldots, i_mj_m)\) be an order of links. At stage \(k\), player \(i_k\) decides whether or not to create link \(i_kj_k\). Her decision becomes publicly known. It is, consequently, a game of perfect information and the argument given by Govindan and McLennan (2001) covers this case.

**Appendix A. Proof of the Proposition**

The current proof is based on the proof of Theorem 5.3 in Govindan and McLennan (2001). It exploits the semi-algebraic structure of the Nash equilibrium correspondence. We will closely follow their exposition of semi-algebraic geometry. Other expositions of the subject in the economic literature can be found in Blume and Zame (1994) and Schanuel et al. (1991). Proofs of major results are omitted.

**Definition A.1.** A set \(A\) is semi-algebraic if it is the finite union of sets of the form

\[
\left\{ x \in \mathbb{R}^m : P(x) = 0 \text{ and } Q_1(x) > 0 \text{ and } \ldots \text{ and } Q_k(x) > 0 \right\}
\]

where \(P\) and \(Q_1, \ldots, Q_k\) are polynomials in \(x_1, \ldots, x_m\) with real coefficients. A function (or correspondence) \(g : A \rightarrow B\) with semi-algebraic domain \(A \subset \mathbb{R}^n\) and range \(B \subset \mathbb{R}^m\) is semi-algebraic if its graph is a semi-algebraic subset of \(\mathbb{R}^{n+m}\).
Each semi-algebraic set is the finite union of connected components. Each component is a semi-algebraic manifold of a given dimension. A d-dimensional semi-algebraic manifold in \(R^n\) is a semi-algebraic set \(M \subset R^n\) such that for each \(p \in M\) there exist polynomials \(P_1, \ldots, P_{m-d}\) and \(U\), a neighborhood of \(p\), such that \(DP_1(p), \ldots, DP_{m-d}(p)\) are linearly independent and
\[
M \cap U = \left\{ q \in U : P_1(q) = \ldots = P_{m-d}(q) = 0 \right\}.
\]

**Theorem A.1** (Stratification, Whitney (1957), Bochnak et al. (1987, 9.1.8)). If \(A\) is a semi-algebraic set, then \(A\) is the union of a finite number of disjoint, connected semi-algebraic manifolds \(A^l\) with \(A^l \subset cl(A^k)\) whenever \(A^l \cap cl(A^k) \neq \emptyset\).

Henceforth, superscripts index members of a decomposition, while subscripts keep indexing players. Theorem A.1 has important consequences. Among those, we will use the following intuitive ones: Let \(A \subset R^m\) and \(B \subset R^n\) be semi-algebraic sets, then

- the dimension of \(A\), \(\dim A\), is equal to the largest dimension of any element of any stratification,
- if \(A\) is 0-dimensional then \(A\) is finite,
- \(A\) is generic if and only if \(\dim(R^m \setminus A) < m\),
- \(\dim(A \times B) = \dim A + \dim B\).

We need one additional result. While Theorem A.1 decomposes semi-algebraic sets, the following one decomposes semi-algebraic functions.

**Theorem A.2** (Generic Local Triviality, Hardt (1980), Bochnak et al. (1987, 9.3.2)). Let \(A\) and \(B\) be semi algebraic sets, and let \(g : A \to B\) be a continuous semi-algebraic function. There is a relatively closed semi-algebraic set \(B^l \subset B\) with \(\dim B^l < \dim B\) such that each component \(B^l\) of \(B \setminus B^l\) has the following property: there is a semi algebraic set \(F^l\) and a semi-algebraic homeomorphism \(h : B^l \times F^l \to A^l\), where \(A^l = g^{-1}(B^l)\), with \(g(h(b, f)) = b\) for all \((b, f) \in B^l \times F^l\).

We can now proceed to prove the Proposition. Recall that at every completely mixed strategy \(\sigma \in \Sigma\), the set of probability distributions on outcomes that player \(i\) can induce by varying her strategy is \(|\{S_i\} - 1|\)-dimensional.

**Proof of the Proposition.** Let \(A = \{(\sigma, u) : \sigma\) is a completely mixed equilibrium for \(u\}\). Let \(\pi_{\Sigma}\) be the projection of \(A\) onto \(\Sigma\). Apply Theorem A.2 to \(\pi_{\Sigma}\) and choose \(\Sigma^l\) such that \(\dim \Sigma^l = \dim A\). We have that \(\dim A = \dim \Sigma^l + \dim F^l \leq \dim \Sigma + \dim F^l\). Let \(\sigma\) belong to \(\Sigma^l\), then \(\dim \pi_{\Sigma}^{-1}(\sigma) = \dim \{\sigma\} + \dim F^l = \dim F^l\).

Now consider a given \(u \in U\). The set consisting of those \(\tilde{u} \in U\) such that \(\sigma\) is a completely mixed equilibrium for \((u, \tilde{u})\) is \((\dim U - (|\{S_i\} - 1|))\)-dimensional. Consequently, the dimension of \(\pi_{\Sigma}^{-1}(\sigma)\) is equal to \(\dim U - \dim \Sigma\), which in turn implies that \(\dim A \leq \dim U\).

Now apply Theorem A.2 to \(\pi_U\), the projection of \(A\) onto \(U\). Choose \(U^l\) to be of the same dimension as \(U\). Therefore, \(\dim A^l = \dim U + \dim \pi_U^{-1}(u)\) for \(u \in U^l\). This implies that \(\dim \pi_U^{-1}(u) \leq \dim A - \dim U \leq 0\), i.e. there is a finite set of completely mixed equilibria whenever \(u\) belongs to a full dimensional \(U^l\). This concludes the proof since lower dimensional \(U^l\)’s are nongeneric. \(\square\)

\textsuperscript{12}Such a \(\Sigma^l\) can be found because we can keep applying Theorem A.2 to \(\pi_{\Sigma} : \pi_{\Sigma}^{-1}(\Sigma^l) \to \Sigma^l\), where \(\Sigma^l\) plays the role of \(B^l\).
References


