VALUE AT RISK OF NON-NORMAL PORTFOLIOS

by

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ABSTRACT

This paper sheds light on the evaluation of portfolio risk when portfolio variables are not normal, as is usually the case with financial variables. The methodology proposed in these cases is based on the assumption of a more general distribution capable of incorporating the behaviour of such variables, especially at the tails: the so called Edgeworth-Sargan distribution. This density is preferable over other distributions, such as the Student’s t, when fitting high frequency financial variables, because of its flexibility for improving data fits by adding more parameters in a natural way.

Furthermore, this distribution is easy to generalise to a multivariate context and, therefore, correlation coefficients among variables can be estimated efficiently. This article, therefore, provides new insights into VaR methodology by estimating the joint density of portfolio variables, and simultaneously calculating the right critical values of the underlying portfolio density. The empirical examples include the estimation and evaluation of different portfolios composed of stock indices for major financial markets.

KEYWORDS
Value at Risk, portfolio distribution, Edgeworth-Sargan density, multivariate distributions.
JEL classification code: G10, G11, C13.
1. INTRODUCTION

Risk Management has become a topic of major interest for both researchers and practitioners over the last decades. Several theories have contributed to this development, such as the portfolio selection models [Markowitz, 1952], the capital asset pricing model [Sharpe, 1964; Lintner, 1965; Mossin, 1966] or the option pricing theory [Black and Scholes, 1973; Merton, 1973], among others. More recently, Value at Risk, or VaR, has experienced a wide development guided by the regulatory requirements for bank solvency [Group of Thirty, 1993, Basle Committee on Banking Supervision, 1995, 1996a,b and 2001]. All these theories represent an overwhelming improvement in quantifying the risk of financial assets, but they still present several shortcomings basically owing to the normality assumption. The non-normality of high frequency financial variables has been widely tested in financial literature [Mandelbrot, 1963; Harvey and Zhoe, 1993; Dacorogma et al., 1995] finding that the density underlying most financial data is more peaked and heavy tailed than the normal distribution.

Many authors have tackled the non-normality of high frequency financial variables from different perspectives ranging from non-parametric [Gallant and Tauchen, 1989; Robinson, 1995; Newley et al., 1998] to parametric estimation of the underlying density, assuming other specifications like the Student’s t [Praetz, 1972; Blattberg and Gonedes, 1974; Rogalski and Vinso, 1978], jump processes [Ball and Torous, 1983; Jorion, 1988], mixtures of normal distributions [Hamilton, 1991; Peiró, 1995], or many other densities [McDonald and Newley, 1988; Baille and Bollerslev, 1990; Mittnik and Rachev, 1993; McDonald and Xu, 1995]. This literature also accounts for the conditional heteroskedasticity phenomenon inherent in this kind of data [Bollerslev, 1987; Baillie and Bollerslev, 1989; Hsieh, 1989;.
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Nelson, 1991; Ding et al., 1993; León and Mora, 1999], the ARCH and GARCH processes by Engle [1982] and Bollerslev [1986] being the most widely used. However, only a few papers focus on the theoretical implications of the non-normality of these variables in financial markets equilibrium – Jurczenko and Maillet [2001] introduce a generalization of CAPM to three moment asset pricing model.

Regarding VaR literature, many different methodologies have been employed to better take into account the underlying density, such as historical simulation data and Monte Carlo experiments or parametric estimation [Jorion, 1997; Coronado, 2000]. Nevertheless, the most popular method – derived by J. P. Morgan Bank [1995] and implemented in “RiskMetrics” – is based on the estimation of the variance and covariance matrix of portfolio assets and relies on the simplified, but probably unreasonable, normality hypothesis. The validity of this hypothesis on calculating VaR has been studied by Finger [1987] and Hull and White [1998], and some alternative specifications have been proposed recently: mixtures of normal distributions [Venkataraman, 1997], Student’s t [Vlaar, 2000; Lucas, 2000], Hyperbolic distributions [Bauer, 2000], switching regime models [Billio and Pellizon, 2000] or semi-parametric techniques [Danielsson and de Vries, 2000].

This article fits into this framework by introducing a new methodology for computing VaR which attempts to solve some of the deficiencies of previous approaches. This paper focuses on the Edgeworth-Sargan distribution – hereafter ES – rather than the normal, which, in fact, is nested on the ES. The Edgeworth-Sargan distribution has been shown capable of capturing salient empirical regularities of the histogram for most high frequency financial variables in Perote [1999] and Mauleón and Perote [2000]. In particular, this distribution can account for thicker tails than the normal, as well as for possible asymmetries, as a result of the
consideration of a general and flexible parameterisation. Parameter flexibility results in improved quality of fit results, compared to other fitted densities – like Student’s t, for instance. Moreover, the ES distribution can be generalised to a multivariate context [Mauleón and Perote, 1999] and thus the variance and covariance matrix of a group of variables can be estimated consistently with the given assumptions. The analytical simplicity of ES represents another advantage of this distribution, since it does not only make optimisation algorithms converge, but also the estimated distribution can be used to calculate probabilities or critical values for confidence intervals, as shown in Perote and Del Brío [2001].

The next section (2) outlines the general model considered for calculating a portfolio’s VaR, and section 3 describes in detail the particular case analysed in the empirical examples. The empirical results (section 4) are drawn from a sample of daily observations, spanning approximately 6 years of stock indices for major financial markets. Finally, conclusions are gathered in section 5.

2. GENERAL PORTFOLIO MODEL

The portfolio VaR describes the worst expected loss of the portfolio in some temporal horizon and at some confidence level. Hence the computation of VaR involves dealing with four main elements: time horizon, confidence level, underlying distribution and portfolio covariances. The time horizon and the choice of the confidence level depend on the risk aversion of the portfolio manager. For example, RiskMetrics recommends the Daily Earning at Risk – hereafter DEaR – whilst Basle Committee shows preference for the ten days period. However, it is clear that bigger time horizons increase the likelihood of parameter instability.
and misleading VaR measures, and the bigger the risk aversion the smaller the confidence level that should be used.

Regarding the portfolio distribution, it is clear that under the normality assumption the portfolio distribution is fully characterised by the variance and covariance matrix, since the whole structure of the multivariate normal only depends on their first and second order moments. Nevertheless, other distributions, such as the ES, depend on more parameters to account for higher order moments. That is probably the main contribution of this paper: the comparison of normality assumption (case A) versus a more general and complex structure (case B) that incorporates other moments which affect the computation of a portfolio’s VaR.

As far as the time varying variance and covariance matrix are concerned, many assumptions may be considered, the ARCH and GARCH models by Engle [1986] and Bollerslev [1987] being the most widespread. In fact, RiskMetrics uses a GARCH(1,1) structure subject to some restricted hypotheses to guarantee the non-stationarity of conditional variances. Nevertheless we do not consider GARCH structures in this article because the Edgeworth-Sargan distribution accounts for thicker tails than the normal capturing the extreme volatility adequately. In what follows we describe the assumptions considered to model portfolio distributions.

Let $P_t$ be the return of some portfolio (see equation 2.1), measured as the weighed sum of the returns of $N$ assets ($r_{it}$). Returns are modelled as the AR(1) processes as it is common in the finance literature. This is shown in 2.2.

$$P_t = \sum_{i=1}^{N} w_i r_{it}, \text{ where } 0 \leq w_i \leq 1 \forall i = 1, 2, ..., N \text{ and } \sum_{i=1}^{N} w_i = 1. \quad [2.1]$$

$$r_{it} = \phi_{0i} + \phi_{1i} r_{i,t-1} + \epsilon_{it}, \quad \forall i = 1, 2, ..., N, \quad [2.2]$$

where $|\phi_{0i}| < 1$ and $\epsilon_i$ stands for a random variable such that $E[r_{it}] = 0$, $E[r_{it}^2] = \sigma_i^2$ and $E[\epsilon_{is}, \epsilon_{j,t-1}] = 0 \forall s \neq 0$ and $\forall t$. 

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Note that, if the returns on all assets are normally distributed (see case A below), the distribution of the portfolio is fully characterised, since normal distribution depends only on their first and second moments and linear combinations of normal variables are normally distributed. Nevertheless we do not necessarily assume that $\varepsilon_i$ is normally distributed (see case B below).

Additionally we define the covariances of returns on assets $i$ and $j$ in terms of their correlation counterparts ($\rho_{ij}$) as follows:

$$\sigma_{ij} = \rho_{ij} \sigma_i \sigma_j$$

where $-1 \leq \rho_{ij} \leq 1$ $\forall i = 1,2,...,N$ and $\forall j = 1,2,...,N$. \[2.3\]

Finally, we consider two distributional hypotheses:

*Case A:*

Let us suppose that $R_i = [r_{1i}, r_{2i}, \ldots, r_{Ni}]$ is multivariate normally distributed, i.e.

$$R_i \sim N(\mu_i, \Sigma), \quad [2.4]$$

where $\mu_i = [\mu_{1i}, \mu_{2i}, \ldots, \mu_{Ni}]$ and $\Sigma = \begin{bmatrix}
\sigma_1^2 & \sigma_{12} & \ldots & \sigma_{1N} \\
\sigma_{12} & \sigma_2^2 & \ldots & \sigma_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{1N} & \sigma_{2N} & \ldots & \sigma_N^2
\end{bmatrix}$. \[2.5\]

($\Sigma$ being a positive definite variance-covariance matrix and $\mu_{it} = \phi_{it} + \phi_i r_{i,t-1}$).

In this case the marginal distributions are univariate normal distributions, denoted by $g_i(r_{it})$ for all $i$. Accordingly, the portfolio distribution can be represented by

$$P_i \sim N(\mu_p, \sigma_p^2)$$

where

$$\mu_p = \sum_{i=1}^{N} w_i \mu_i$$

and

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} w_i w_j \rho_{ij} \sigma_i \sigma_j$$

\[2.7\]
A straightforward generalisation of this model to non-normality requires the calculation of higher order moments also affecting VaR. This is taken up in case B.

Case B:

Let \( r_{it} \) be distributed as Edgeworth-Sargan, hereafter \( r_{it} \sim ES(\mu_{it}, \sigma_i^2) \). Densities, for the returns of each item in the portfolio are expressed in terms of (2.8).

\[
f_i(r_{it}) = g_i(r_{it})\left\{1 + q_i(v_{it})\right\}
\]  

[2.8]

where \( g_i(r_{it}) \) is \( N(\mu_{it}, \sigma_i^2) \) and \( q_i(v_{it}) = \sum_{j=3}^{\infty} d_{ij} H_j(v_{it}) \) is a linear combination of the Hermite polynomials \( H_j(v_{it}) \), \( d_{ij} \) being the parameter of each variable \( v_{it} = \frac{r_{it} - \mu_{it}}{\sigma_i} \) and for each polynomial \( s \). These Hermite polynomials fulfil the following identity:

\[
\frac{d^j g(v_{it})}{dv_{ij}^j} = (-1)^j g(v_{it}) H_j(v_{it})
\]  

[2.9]

where \( g(v_{it}) \) is a standard normal density, i.e. \( N(0,1) \).

Following Kendall and Stuart [1977] Hermite polynomials can be obtained by (2.10).

\[
H_j(v_u) = v_u^j - \frac{j(j-1)}{2} v_u^{j-2} + \frac{j(j-1)(j-2)(j-3)}{2!3!} v_u^{j-4} - \frac{j(j-1)(j-2)(j-3)(j-4)(j-5)}{2!3!4!} v_u^{j-6} + \ldots
\]  

[2.10]

Mauleón and Perote [1999] show that the Edgeworth-Sargan distribution can be easily generalised to the multivariate case. These authors provided an expression for the multivariate Edgeworth-Sargan distribution which satisfies an interesting property: its marginal distributions behave also as Edgeworth-Sargan. Therefore, we use this representation to account for the joint behaviour of portfolio variables under case B, i.e.
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\[ R_i \sim ES(\mu_i, \Sigma), \]  

where the joint density of the vector \( R \) is

\[ F(R) = G(R) + \left[ \prod_{i=1}^{N} g_i(r_i) \right] \left[ \sum_{i=1}^{N} q_i(v_i) \right], \]

and \( G(R) \) stands for the multivariate normal density shown in (2.13).

\[ G(R) = (2\pi)^{-N/2} |\Sigma|^{-1/2} \exp\left\{ -\frac{1}{2} (R - \mu^T \Sigma^{-1} (R - \mu) \right\} \]

It is easy to check that linear combinations of Edgeworth-Sargan distributions are also Edgeworth-Sargan distributed – see Kendall and Stuart [1977] for further details on Hermite and Edgeworth expansions and their properties –, and that the portfolio is:

\[ P \sim ES(\mu_p, \sigma_p^2) \]

where \( \mu_p \) and \( \sigma_p^2 \) satisfy (2.7).

3. CALCULATING VaR FOR A SIMPLE PORTFOLIO

For the sake of clarity, this section firstly describes the kind and composition of the portfolios employed for empirical analysis in the next section, i.e. two-asset portfolios. Therefore, let \( P \) be a linear convex combination of \( r_{1t} \) and \( r_{2t} \), which might represent returns on stock market indices or treasury bond yields, for example. Hence,

\[ P_t = w_1 r_{1t} + w_2 r_{2t}, \text{ where } 0 \leq w_i \leq 1, \ \forall i = 1, 2 \text{ and } w_1 + w_2 = 1. \]

Let us also assume that the conditional mean and the variance of both assets can be represented by \( \mu_i \) and \( \sigma_i^2 \) and \( \mu_2 \) and \( \sigma_2^2 \), respectively, and the correlation coefficient is \( \rho_{12} \). Moreover, for further analysis of portfolio risk, it is necessary to assume a distributional
hypothesis. Once more, case A reflects the commonly used normality assumption, whilst case B imposes a more complex structure based on the Edgeworth-Sargan distribution.

**Case A:**

The joint distribution of $r_{1t}$ and $r_{2t}$ is given in (3.3), and the distribution of the portfolio is derived in (3.4).

$$G(r_{1t}, r_{2t}) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1 - \rho_{12}^2}} \exp \left\{ -\frac{1}{2} \left( \frac{r_{1t} - \mu_{1t}}{\sigma_1} \right)^2 + \left( \frac{r_{2t} - \mu_{2t}}{\sigma_2} \right)^2 - 2\rho_{12} \frac{(r_{1t} - \mu_{1t})(r_{2t} - \mu_{2t})}{\sigma_1 \sigma_2} \right\} \right. $$

[3.3]

$$g_p(P_i) = \frac{1}{\sigma_p \sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left( \frac{P_i - \mu_{pt}}{\sigma_p} \right)^2 \right\} \right. $$

[3.4]

It is also straightforward that the portfolio conditional mean and variance are

$$\mu_{pt} = w_1 \mu_{1t} + w_2 \mu_{2t} \quad \text{and} \quad \sigma^2_p = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1w_2 \rho_{12} \sigma_1 \sigma_2 $$

[3.5]

Under all of these assumptions the VaR of the portfolio for next period at confidence level $\alpha$ may be easily computed as indicated in (3.6).

$$VaR = \sigma_p \phi(\alpha) \quad \text{where} \quad \alpha = \Pr \{ x \leq \phi(\alpha) \} \quad \text{if} \quad x \sim N(0,1). $$

[3.6]

**Case B:**

The joint distribution of $r_{1t}$ and $r_{2t}$, and the portfolio density are shown in (3.7) and (3.9), respectively.

$$F(r_{1t}, r_{2t}) = G(r_{1t}, r_{2t}) + g_1(r_{1t})g_2(r_{2t})\left[ q_1(v_{1t}) + q_2(v_{2t}) \right] $$

[3.7]

where
\begin{equation}
g_i(r_{it}) = \frac{1}{\sigma_i \sqrt{2\pi}} \exp\left\{-\frac{1}{2} \left( \frac{r_{it} - \mu_{it}}{\sigma_i} \right)^2 \right\} \quad \text{and} \quad q_i(r_{it}) = \sum_{s=3}^{S_i} d_{si} H_i \left( \frac{r_{it} - \mu_{it}}{\sigma_i} \right) \forall i = 1, 2.
\end{equation}

\[3.8\]

\begin{equation}
f_p(P_i) = g_p(P_i) \left\{ 1 + \sum_{s=3}^{S_p} d_{sp} H_s \left( \frac{P_i - \mu_{pt}}{\sigma_p} \right) \right\}.
\end{equation}

\[3.9\]

It must be noticed that \( \mu_{pt} \) and \( \sigma_p^2 \) satisfy (3.5) and \( g_p(P_i) \) is stated in (3.4).

Moreover, the VaR of the portfolio for next period and at confidence level \( \alpha \) may be obtained from (3.6), assuming that \( \phi(\alpha) \) stands for the critical value of a standard Edgeworth-Sargan density, i.e. \( x \sim ES(0,1) \). In fact, this is the main advantage of this approach, since the Edgeworth-Sargan distribution can account for the thick tails of high frequency financial data and, therefore, measure the probability of extreme values more accurately. The only problem seemed to lie in calculating these critical values, however, it does appear to be a problem any longer, since the Edgeworth-Sargan distribution is easy to integrate. Actually, if we suppose that \( x \sim ES(0,1) \) with pdf \( f(\cdot) \) and \( g(\cdot) \) stand for the \( N(0,1) \) density, it can be proved that

\[ \Pr \{ x \leq \phi(\alpha) \} = \int_{-\infty}^{\phi(\alpha)} f(x) dx = \int_{-\infty}^{\phi(\alpha)} g(x) dx - g(\phi(\alpha)) \sum_{s=2}^{S} d_s H_{s-1}(\phi(\alpha)). \]

\[3.10\]

See the Appendix for the demonstration.

4. EMPIRICAL RESULTS

This section compares different portfolios’ VaRs for different confidence levels and weights. In particular, the models shown in the previous section are estimated by maximum likelihood and the expected loss for next period – DEaR approach is assumed – is measured as a percentage of the investment and at different confidence level. For the sake of
comparison, both the multivariate normal and the ES are estimated. Notice that under normal distribution, VaR can be computed directly through the estimated variance and covariance matrix but, on the other hand, if the ES distribution is assumed the rest of the parameters of the portfolio distribution must be estimated before VaR is calculated, and thus the exact critical value or quantile.

Therefore, the methodology used in this section for calculating a portfolio VaR can be summarised in the following steps: firstly, the conditional multivariate distribution for portfolio variables is jointly estimated (either under normality or ES assumption). Secondly, the conditional ES density of the portfolio is estimated, subject to the estimates of the variance and covariance matrix obtained in the first step. Finally, the accurate critical value of the estimated ES for the portfolio – at each confidence level – is computed. A portfolio VaR is just the product of this critical value and the estimated deviation for the portfolio. VaRs under both specifications are compared for different confidence levels (0.05, 0.025 and 0.01). As a consequence of the ES tails being thicker than those of the normal distribution, the higher the confidence level, the larger the expected difference in VaR between both approaches.

Tables 1 and 2 show the VaR estimates for different portfolios and scenarios. In particular, Table 1 displays the estimates for some portfolios composed of the General Madrid Stock Exchange index – IGBM vi hereafter – and the Dow Jones index using different weights: 0.1, 0.5 and 0.9. Table 2 summarizes the results for some portfolios composed of the IGBM and the FTSE index. For the sake of clarity, parameter notation follows that of the previous section vii. The concrete daily series used in empirical work are the general Madrid
Stock index and the Dow Jones industrial price index and the FTSE total non-financial price index ranging from 2/1/90 to 28/5/96.

Turning to the empirical results shown in the tables, an asterisk indicates the non-significant parameters at 5 percent confidence level. The log likelihood value of each density is also displayed, they reflect the fact that the multivariate conditional ES distribution is clearly preferred to the multivariate conditional normal to represent the joint behaviour of portfolio variables. This evidence reinforces the fact of the non-normality of these variables and, therefore, the more efficient estimation of portfolio density obtained by the more general ES distribution. Once this hypothesis is tested, a sensitivity analysis is performed, showing how VaR is influenced by the wrong normality assumption for different portfolios and confidence levels.

The results clearly conclude that for high confidence levels (0.05) there exists an overestimation of the true VaR when accounting only for the first and second moments (i.e. under the normality assumption). Nevertheless the more risk averse the agents are (high confidence levels) the bigger the underestimation of VaR that is found. Actually the VaR understatement goes up to 75% when the traditional normal model is used and at one percent confidence level. This evidence was also found by Lucas [2000] when comparing usual VaRs to these obtained under a Student’s t distribution. Moreover, this result is also in line with Danielsson and de Vries [2000], who assessed that at 5 percent confidence level RiskMetricks is adequate but underpredicts VaR compared to a semi-parametric method, which is more accurate capturing portfolios’ tails behaviour.

FIGURES 1 and 2 highlight the superior performance of the estimate of the portfolio density assuming the Edgeworth-Sargan distribution compared to the estimate under the
normal distribution. The empirical histogram of the data is also plotted revealing that the ES distribution is capturing reasonably well the data behaviour even at the tails. In particular the portfolio composed of IGBM and Dow Jones indices for \( w_1=0.5 \) are displayed, thus from Figure 2 it is easy to realise the underestimation and overestimation of VaR of the normal density (compared to the ES fitted density) for values in the interval \((-0.02; -0.015)\).

In conclusion, the use of the ES distribution on calculating portfolio VaR is easy to be implemented and leads to more accurate VaR measures. Additionally, the model presented on this paper can be easily generalised to incorporate different conditional variance measures – provided by GARCH models, for example – and even the portfolio weights could be interpreted as parameters of the model and therefore optimal weighting policies could be derived from this approach. However, the introduction of more complex structures to the model could impose identification or computational problems that should be solved by constraining some parameter of the model or selecting adequately the initial values for the optimisation algorithms.

5. CONCLUSIONS

The Edgeworth-Sargan density has been shown capable of fitting financial univariate densities as well as multivariate densities in previous articles [Mauleón and Perote, 1999; 2000]. However, unless this distribution were able to capture VaR adequately, its usefulness would be jeopardized, somehow. The purpose of this paper is to show the applicability of this distribution to measure VaR more accurately than through the traditional normal assumption. In this paper, a methodology for computing VaR consistently with the ES distributional hypothesis has been developed and applied to calculate VaR for different portfolios at
different confidence levels. For the sake of comparison, VaR under normal hypothesis has also been computed, hence, the differences between both methodologies represent clear evidence of an understatement of VaR in most cases, especially for low confidence levels.

The main facts, supporting the claims of the preceding paragraph, can be summarized as follows:

1.- High frequency financial variables are found to be far from the normality assumption, therefore traditional VaR methodologies should be modified accordingly to this evidence. Edgeworth-Sargan distribution is able to account for the main empirical features of the underlying density and provides a flexible and simple parametric representation to incorporate into VaR methodology. Moreover, the normal distribution is nested on the Edgeworth-Sargan and thus testing and comparing both distributional hypotheses can be easily analysed.

2.- The variance and covariance matrix of portfolio variables may be estimated consistently with the Edgeworth-Sargan hypothesis. Moreover, portfolio Edgeworth-Sargan distribution can be easily estimated subject to the previous estimation of portfolio variance. Despite not being the main purpose of this paper, different time varying variance hypotheses could have been also implemented. The computational problems of this complex approach might be solved by accurately selecting the initial values according to the more simple Edgeworth-Sargan structure of the portfolio variables.

3.- VaRs obtained assuming an Edgeworth-Sargan appear to be more similar (even smaller) to VaR under normality hypothesis for some portfolios at high confidence levels. Nevertheless, in most cases, the portfolio VaR under ES distribution is clearly bigger. This VaR understatement is especially highlighted for low confidence levels.
4.- This general approach to calculate VaR assuming an ES distribution can help risk managers to accurately measure risk, since the risk measurements can be improved by adding more parameters to the flexible ES parametric formulation. Moreover, optimal weighting policies can be also straightforward developed.

APPENDIX

We shall prove that \( \int f(x)dx = \int g(x)dx - g(x) \sum_{s=2}^{S} d_s H_{s-1}(x) \), where \( g(x) \) stands for \( N(0,1) \).

Proof:

\[
\int f(x)dx = \int g(x) \left[ 1 + \sum_{s=2}^{S} d_s H_s(x) \right] dx = \int g(x)dx + \sum_{s=2}^{S} d_s (-1)^s \int (-1)^s g(x)H_s(x)dx = \]

\[
= \int g(x)dx - g(x) \sum_{s=2}^{S} d_s H_{s-1}(x) , \] as a direct result of 2.9.

Q.E.D.

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* The parameter is not significant at 5% confidence level.

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<th>$w_1 = .5$</th>
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<td>0.26 x $10^{-3}$</td>
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* The parameter is not significant at 5% confidence level.
REFERENCES


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1 However, this kind of volatility models can be introduced to the methodology employed in this paper in a straightforward manner.

2 Note that the normal distribution is a particular case of the Edgeworth-Sargan distribution.

3 For the sake of clarity, this notation is used despite the fact that Edgeworth-Sargan distribution does not depend on its first and second moments only.

4 In particular, the polynomials used in section 4 are just $H_4(v_{it}) = v_{it}^4 - 6v_{it}^2 + 3$, $H_6(v_{it}) = v_{it}^6 - 15v_{it}^4 + 45v_{it}^2 - 15$ and $H_8(v_{it}) = v_{it}^8 - 28v_{it}^6 + 210v_{it}^4 - 420v_{it}^2 + 105$.

5 This study could be extended to a multidimensional context by considering multi-asset portfolios, although the complexity of the model proposed would dramatically increase. Therefore, to avoid potential parameter identification problems, we have only considered two-asset portfolios. However, the consideration of multi-asset portfolios would likely diminish deviations from normality.

6 Indice General de la Bolsa de Madrid.

7 It is worth to note that some of the parameters of the ES distribution have been constrained to zero after testing these hypotheses. In particular, no evidence of asymmetries were found (contrary to other authors like Theodosiou, 1998), and therefore all the odd parameters of the ES distribution were cancelled out.

8 As the multivariate normal is nested on the multivariate ES, a simple LR test can be implemented straightforward to reject the normality assumption.

9 In this sense, it is remarkable the evidence found by Perote (1999) and Mauleón and Perote (2000) showing that the ES distribution is able to capture the behaviour of high frequency financial variables even better than other simpler parametrical structures, such as the Student’s $t$.

10 Notice that the use of a general ES structure can also be interpreted as a semi-non-parametric approach to the underlying density.