Symmetry reductions of a $2 + 1$ Lax pair

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The identification of the Lie symmetries of a PDE is an instrument of primary importance in order to solve such an equation. A standard method for finding solutions of PDEs is that of reduction using Lie symmetries:

Each Lie symmetry allows a reduction of the PDE to a new equation with the number of independent variables reduced by one

this procedure gives rise to the ARS conjecture which establishes that a PDE is integrable in the Painlevé sense if all its reductions pass the Painlevé test.

This means that solutions of a PDE can be achieved by solving its reductions to ordinary differential equations (ODE).

Classical and non-classical (Olver, Bluman, Cole) Lie symmetries are the usual way for identifying the reductions.

Nevertheless, let us recall that there are some methods for solving PDEs that are more effective in $2 + 1$ than in $1 + 1$ dimensions.

- The Singular Manifold Method (Weiss) based on the Painlevé property requires the truncation of the Painlevé series at the constant level. Sometimes this condition is very restrictive and must be modified (Estevez-Leble 1995).
- For some equations in $1 + 1$ dimensions the SMM imposes so many restrictions that there is not freedom enough to get nontrivial solutions and/or to introduce a spectral parameter (Estevez et al 1993).
- Nevertheless, these problems disappear when we pass to higher dimensions. It is necessary to increase the number of dimensions in order to have sufficient freedom so that the SMM will not be too restrictive.

Equation

A good example of the above statement is the following equation in $2 + 1$ dimensions

$$\left[h_{xxz} - \frac{3}{4} \left(\frac{h_{xz}^2}{h_z} \right) + 3h_x h_z \right]_x = h_{yz},$$

which we have studied in a recent paper (JNMP 2005) proving that the SMM is a very effective method for solving the equation. It is quite straightforward to determine the associated linear problem through the SMM.

$$\begin{aligned} -\psi_y + \psi_{xxx} + 3h_x \psi_x + \frac{3}{2} h_{xx} \psi &= 0, \\ 2h_z \psi_{xz} - h_{xz} \psi_z + 2h_z^2 \psi &= 0, \end{aligned}$$

The SMM also provides Darboux transformations and an iterative method to construct solutions. Nevertheless, different reductions of our equation) generate a lot of problems when we attempt to apply the SMM to them (JNMP 2005).

This means that the SMM allows us to solve the PDE in $2 + 1$ dimensions but not their reductions to $1 + 1$.

Therefore, our plan in this paper is, to a certain extent, exactly the opposite of the usual approach: we try to obtain $1+1$ spectral problems arising from a $2 + 1$ Lax pair.

- Once we have solved the problem in $2 + 1$ dimensions, in the sense that we have determined its Lax pair, we shall identify the classical symmetries of the Lax pair.
- We use these symmetries to obtain a reduced Lax pair in $1 + 1$ dimensions whose compatibility condition should be a reduction of the equation. The $2 + 1$ Lax pair does not contain a spectral parameter but the reduction process should introduce a spectral parameter that is absolutely essential in $1 + 1$ dimensions.
- Actually, there are five possible reductions. Three of them yield linear equations that can be easily integrated. The other two reductions yield $1 + 1$ spectral problems that include, well known equations.
- It is interesting to note that each of these reductions yields respectively two, three and fourth order spectral problems.

Classical Symmetries

In order to apply the **Classical Lie Method** to the system of PDEs with three independent variables and two fields, we consider the one-parameter following Lie group of infinitesimal transformations in x, y, z, h, ψ :

$$\begin{aligned}x' &= x + \varepsilon \xi_1(x, y, z, h, \psi) + O(\varepsilon^2) \\y' &= y + \varepsilon \xi_2(x, y, z, h, \psi) + O(\varepsilon^2) \\z' &= z + \varepsilon \xi_3(x, y, z, h, \psi) + O(\varepsilon^2) \\h' &= h + \varepsilon \phi_1(x, y, z, h, \psi) + O(\varepsilon^2) \\\psi' &= \psi + \varepsilon \phi_2(x, y, z, h, \psi) + O(\varepsilon^2)\end{aligned}$$

where ε is the group parameter. It is therefore necessary that this one transformation leaves the set of solutions invariant. **This yields an overdetermined linear system of equations for the infinitesimals $\xi_1, \xi_2, \xi_3, \phi_1$ and ϕ_2 .**

The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form:

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \xi_3 \frac{\partial}{\partial z} + \phi_1 \frac{\partial}{\partial h} + \phi_2 \frac{\partial}{\partial \psi}$$

By applying the classical method, we obtain the following generators:

$$\xi_1 = -2 \frac{dA_1(y)}{dy} x - \frac{3}{2} A_2(y)$$

$$\xi_2 = -6A_1(y)$$

$$\xi_3 = \beta(z)$$

$$\phi_1 = \frac{1}{3} \frac{d^2 A_1(y)}{dy^2} x^2 + \frac{1}{2} \frac{dA_2(y)}{dy} x + A_3(y) + 2 \frac{dA_1(y)}{dy} h$$

$$\phi_2 = \left(\lambda + \frac{dA_1(y)}{dy} \right) \psi$$

The corresponding Lie symmetry algebra depends on three $A_1(y)$, $A_2(y)$ and $A_3(y)$ arbitrary functions of y and an arbitrary function $\beta(z)$ of z . The only constant that appears is λ that, as we shall show, plays the role of the spectral parameter in the $1 + 1$ reductions.

Having determined the infinitesimals, the symmetry variables are found by solving the invariant surface conditions

$$\Phi_1 \equiv \xi_1 \frac{\partial h}{\partial x} + \xi_2 \frac{\partial h}{\partial y} + \xi_3 \frac{\partial h}{\partial z} - \phi_1 = 0$$

$$\Phi_2 \equiv \xi_1 \frac{\partial \psi}{\partial x} + \xi_2 \frac{\partial \psi}{\partial y} + \xi_3 \frac{\partial \psi}{\partial z} - \phi_2 = 0$$

or the corresponding characteristic equations

$$\begin{aligned} \frac{-dx}{2\frac{dA_1}{dy}x + \frac{3}{2}A_2} &= \frac{-dy}{6A_1} = \frac{dz}{\beta} = \\ &= \frac{dh}{\frac{1}{3}\frac{d^2A_1}{dy^2}x^2 + \frac{1}{2}\frac{dA_2}{dy}x + A_3 + 2\frac{dA_1}{dy}h} = \frac{d\psi}{\left(\lambda + \frac{dA_1}{dy}\right)\psi} \end{aligned}$$

There are different possibilities.

Reductions

There are five independent reductions that we determine in the following way

Case 1) $A_1 \neq 0, \quad \beta \neq 0$

The reduced variables are :

$$z_1 = \frac{x}{A_1^{1/3}} - \frac{1}{4} \int \frac{A_2}{A_1^{4/3}} dy, \quad z_2 = 6 \int \frac{1}{\beta} dz + \int \frac{1}{A_1} dy$$

and the following reduction for the fields

$$\begin{aligned} \psi(x, y, z) &= \frac{e^{-\frac{\lambda}{6} \int \frac{dy}{A_1}}}{A_1^{1/6}} G(z_1, z_2) \\ h(x, y, z) &= \frac{1}{A_1^{1/3}} U(z_1, z_2) - \frac{1}{6A_1^{1/3}} \int A_1^{1/3} M(z_1, y) dy \end{aligned}$$

where $M(z_1, y)$ is the function

$$\begin{aligned}
 M(z_1, y) = & \left(\frac{d^2 A_1}{dy^2} \frac{1}{3 A_1^{1/3}} \right) z_1^2 + \\
 & + \left(\frac{dA_2}{dy} \frac{1}{2 A_1^{2/3}} + \frac{d^2 A_1}{dy^2} \frac{1}{6 A_1^{1/3}} \int \frac{A_2}{A_1^{4/3}} dy \right) z_1 + \\
 & + \frac{d^2 A_1}{dy^2} \frac{1}{48 A_1^{1/3}} \left(\int \frac{A_2}{A_1^{4/3}} dy \right)^2 + \frac{dA_2}{dy} \frac{1}{8 A_1^{2/3}} \int \frac{A_2}{A_1^{4/3}} dy + \frac{A_3}{A_1}
 \end{aligned}$$

Substitution of the reduction ansatz gives us:

$$\begin{aligned}
 0 &= G_{z_1 z_1 z_1} - G_{z_2} + 3U_{z_1} G_{z_1} + \frac{3}{2} U_{z_1 z_1} G + \frac{\lambda}{6} G \\
 0 &= 2U_{z_2} G_{z_2 z_1} - U_{z_2 z_1} G_{z_2} + 2U_{z_2}^2 G
 \end{aligned}$$

Solving for G_{z_2} , we obtain **the fourth order spectral problem**:

$$\begin{aligned}
 G_{z_2} &= G_{z_1 z_1 z_1} + 3U_{z_1} G_{z_1} + \frac{3}{2} U_{z_1 z_1} G + \frac{\lambda}{6} G \\
 U_{z_2} G_{z_1 z_1 z_1 z_1} &= \frac{1}{2} U_{z_2 z_1} G_{z_1 z_1 z_1} - 3U_{z_1} U_{z_2} G_{z_1 z_1} + \\
 &+ \left(\frac{3}{2} U_{z_1} U_{z_2 z_1} - \frac{\lambda}{6} U_{z_2} - \frac{9}{2} U_{z_1 z_1} U_{z_2} \right) G_{z_1} + \\
 &+ \left(\frac{\lambda}{12} U_{z_2 z_1} - U_{z_2}^2 - \frac{3}{2} U_{z_1 z_1 z_1} U_{z_2} + \frac{3}{4} U_{z_1 z_1} U_{z_2 z_1} \right) G
 \end{aligned}$$

whose compatibility condition gives us the 1 + 1 equation

$$\left(U_{z_2 z_1 z_1} - \frac{3}{4} \frac{U_{z_2 z_1}^2}{U_{z_2}} + 3U_{z_2} U_{z_1} \right)_{z_1} = U_{z_2 z_2}$$

that can be written as the **Drinfel'd Sokolov nonlocal KdV equation**

Case 2) $A_1 \neq 0, \quad \beta = 0$

$$z_1 = \frac{x}{A_1^{1/3}} - \frac{1}{4} \int \frac{A_2}{A_1^{4/3}} dy, \quad z_2 = z$$

and the reduction for the fields is exactly the same as in case 1. We obtain **the third order spectral problem**:

$$0 = G_{z_1 z_1 z_1} + 3U_{z_1} G_{z_1} + \frac{3}{2} U_{z_1 z_1} G + \frac{\lambda}{6} G$$

$$0 = 2U_{z_2} G_{z_2 z_1} - U_{z_2 z_1} G_{z_2} + 2U_{z_2}^2 G$$

whose compatibility gives us the 1 + 1 equation

$$\left(U_{z_2 z_1 z_1} - \frac{3}{4} \frac{U_{z_2 z_1}^2}{U_{z_2}} + 3U_{z_2} U_{z_1} \right)_{z_1} = 0$$

An alternative form of the above equation arises from the following definitions

$$U_{z_1} = \frac{4}{3}V, \quad U_{z_2} = -P^2$$

which allow us to write it as the system

$$PP_{z_1 z_1} - \frac{P_{z_1}^2}{2} + 2P^2 V + F(z_2) = 0, \quad V_{z_2} = -\frac{3}{4}(P^2)_{z_1}$$

where $F(z_2)$ is an arbitrary function. It is the **Ermakov-Pinney equation** that is related through a reciprocal transformation to the Degasperis-Procesi equation.

Case 3) $A_1 = 0, \quad A_2 \neq 0, \quad \beta \neq 0$

$$z_1 = x + \frac{3}{2}A_2 \int \frac{1}{\beta} dz, \quad z_2 = y$$

$$\psi(x, y, z) = e^{\left(-\frac{2\lambda}{3A_2}\right)x} G(z_1, z_2)$$

$$h(x, y, z) = U(z_1, z_2) - \frac{1}{6} \frac{d^2 A_2}{A_2} x^2 - \frac{2A_3}{3A_2} x$$

Substitution of the reduction ansatz gives us:

$$0 = G_{z_1 z_1 z_1} - G_{z_2} - \frac{2\lambda}{A_2} G_{z_1 z_1} + \left(3U_{z_1} + \frac{4\lambda^2}{3A_2^2} - \frac{2A_3}{A_2} - \frac{dA_2}{dz_2} \frac{z_1}{A_2} \right) G_{z_1}$$

$$+ \left(\frac{3}{2} U_{z_1 z_1} - \frac{2\lambda}{A_2} U_{z_1} + \frac{4\lambda A_3}{3A_2^2} - \frac{8\lambda^3}{27A_2^3} - \frac{1}{2A_2} \frac{dA_2}{dz_2} \right) G$$

$$0 = U_{z_1} G_{z_1 z_1} - \left(\frac{1}{2} U_{z_1 z_1} + \frac{2\lambda}{3A_2} U_{z_1} \right) G_{z_1} + U_{z_1}^2 G$$

Solving for $G_{z_1 z_1}$, we obtain **the second order linear system**:

$$\begin{aligned}
 G_{z_1 z_1} &= \left(\frac{2\lambda}{3A_2} + \frac{U_{z_1 z_1}}{2U_{z_1}} \right) G_{z_1} - U_{z_1} G \\
 G_{z_2} &= \left(\frac{4\lambda^2}{9A_2^2} - \frac{U_{z_1 z_1}^2}{4U_{z_1}^2} - \frac{\lambda}{3A_2} \frac{U_{z_1 z_1}}{U_{z_1}} - \frac{z_1}{A_2} \frac{dA_2}{dz_2} \right) G_{z_1} + \\
 &+ \left(\frac{U_{z_1 z_1 z_1}}{2U_{z_1}} + 2U_{z_1} - 2\frac{A_3}{A_2^2} \right) G_{z_1} + \\
 &+ \left(\frac{4\lambda A_3}{3A_2^2} - \frac{2\lambda}{3A_2} U_{z_1} - \frac{8\lambda^3}{27A_2^3} - \frac{1}{2A_2} \frac{dA_2}{dz_2} \right) G
 \end{aligned}$$

whose compatibility condition yields to the 1 + 1 equation

$$\left(U_{z_2} - U_{z_1 z_1 z_1} + \frac{3 U_{z_1 z_1}^2}{4 U_{z_1}} - 3 U_{z_1}^2 + \frac{2 A_3}{A_2} U_{z_1} + \frac{(z_1 U_{z_1} + U) dA_2}{A_2 dz_2} \right)_{z_1} = 0$$

Setting

$$U_{z_1} = - \left(\frac{Q(T, X)}{A_2} \right)^2, \quad T = \int \frac{1}{A_2^3} dz_2, \quad X = \frac{z_1}{A_2} - 2 \int \frac{A_3}{A_2^2} dz_2$$

we have the **Modified Korteweg-de Vries equation**

$$Q_T - Q_{XXX} + 6Q^2 Q_X = 0$$

Case 4) $A_1 = 0, \quad A_2 \neq 0, \quad \beta = 0$

$$\begin{aligned} z_1 &= y, & z_2 &= z \\ \psi(x, y, z) &= e^{-\frac{2\lambda}{3A_2}x} G(z_1, z_2) \\ h(x, y, z) &= U(z_1, z_2) - \frac{1}{6A_2} \frac{dA_2}{dy} x^2 - \frac{2A_3}{3A_2} x \end{aligned}$$

whose substitution gives us

$$G_{z_1} = \left(\frac{4\lambda A_3}{3A_2^2} - \frac{8\lambda^3}{27A_2^3} - \frac{1}{A_2} \frac{dA_2}{dz_1} \right) G, \quad G_{z_2} = \frac{3A_2}{2\lambda} U_{z_2} G$$

whose compatibility provides the linear equation

$$A_2 U_{z_1 z_2} + \frac{dA_2}{dz_1} U_{z_2} = 0 \implies U(z_1, z_2) = \frac{K_1(z_2)}{A_2} + F_1(z_1)$$

where $F_1(z_1)$ and $K_1(z_2)$ are arbitrary functions. We can integrate as:

$$G = \alpha_0 \frac{1}{\sqrt{A_2}} e^{\frac{3K_1}{2\lambda} + \frac{4\lambda}{3} \int \frac{A_3}{A_2^2} dz_1 - \frac{8\lambda^3}{27} \int \frac{1}{A_2^3} dz_1}$$

where α_0 is an arbitrary constant. The resulting solution is

$$h = \frac{K_1}{A_2} + F_1 - \frac{1}{6A_2} \frac{dA_2}{dy} x^2 - \frac{2A_3}{3A_2} x$$

$$\psi = \frac{\alpha_0}{\sqrt{A_2}} e^{\frac{3K_1}{2\lambda} + \frac{4\lambda}{3} \int \frac{A_3}{A_2^2} dy - \frac{8\lambda^3}{27} \int \frac{1}{A_2^3} dy - \frac{2\lambda}{3A_2} x}$$

depending on two arbitrary constants α_0 and λ and five arbitrary functions $A_1(y)$, $A_2(y)$, $A_3(y)$, $F_1(y)$, and $K_1(z)$.

Case 5) $A_1 = 0, \quad A_2 = 0, \quad A_3 \neq 0, \quad \beta \neq 0$

$$z_1 = x, \quad z_2 = y$$

$$\psi(x, y, z) = e^{\lambda \int \frac{dz}{\beta}} G(z_1, z_2), \quad h(x, y, z) = U(z_1, z_2) + A_3 \int \frac{dz}{\beta}$$

Substitution of the reduction ansatz gives us:

$$0 = G_{z_1 z_1 z_1} - G_{z_2} + 3U_{z_1} G_{z_1} + \frac{3}{2} U_{z_1 z_1} G, \quad 0 = \lambda G_{z_1} + A_3 G$$

The compatibility condition yields to the linear equation

$$\lambda U_{z_1 z_1 z_1} - 2A_3 U_{z_1 z_1} + \frac{2dA_3}{3dz_2} = 0$$

that can be easily integrated as:

$$\begin{aligned} U(z_1, z_2) = & F_3(z_2) e^{\frac{2A_3}{\lambda} z_1} + \frac{1}{6A_3} \frac{dA_3}{dz_2} z_1^2 + \left(\frac{\lambda}{6A_3^2} \frac{dA_3}{dz_2} + \frac{1}{2A_3} F_1(z_2) \right) z_1 \\ & + \frac{\lambda^2}{12A_3^3} \frac{dA_3}{dz_2} + \frac{\lambda}{4A_3^2} F_1(z_2) + \frac{1}{2A_3} F_2(z_2) \end{aligned}$$

where $F_1(z_2)$, $F_2(z_2)$ and $F_3(z_2)$ are arbitrary functions.

$$G = \alpha_0 e^{-\left(\frac{3}{2\lambda} \int F_1 dz_2 + \frac{1}{\lambda^3} \int A_3^3 dz_2 + \frac{A_3}{\lambda} z_1\right)}$$

where α_0 is an arbitrary constant. We obtain the following solution:

$$\begin{aligned} h &= F_3 e^{\frac{2A_3}{\lambda}x} + \frac{1}{6A_3} \frac{dA_3}{dy} x^2 + \left(\frac{\lambda}{6A_3^2} \frac{dA_3}{dy} + \frac{1}{2A_3} F_1 \right) x \\ &+ \frac{\lambda^2}{12A_3^3} \frac{dA_3}{dy} + \frac{\lambda}{4A_3^2} F_1 + \frac{1}{2A_3} F_2 + A_3 \int \frac{dz}{\beta} \\ \psi &= \alpha_0 e^{\lambda \int \frac{dz}{\beta} - \left(\frac{3}{2\lambda} \int F_1 dy + \frac{1}{\lambda^3} \int A_3^3 dy + \frac{A_3}{\lambda} x \right)} \end{aligned}$$

depending on two arbitrary constants α_0 and λ and six arbitrary functions $A_1(y)$, $A_2(y)$, $A_3(y)$, $F_1(y)$, $F_2(y)$, $F_3(y)$ and $\beta(z)$.

Conclusions

- A spectral problem in $2 + 1$ dimensions is presented. It should be noted that this Lax pair was obtained using the SMM that, surprisingly, does not work properly for some $1 + 1$ reductions of the system. It is easier to solve the problem in $2 + 1$ than in $1 + 1$ dimensions
- For the above reason we attempt to go from $2 + 1$ to $1 + 1$ dimensions by using the reductions arising from the Classical Lie symmetries of the Lax pair. This means that we obtain symmetries that are symmetries of both the field h and the eigenfunction ψ . The symmetries that we have obtained include several arbitrary functions as well as an arbitrary constant that plays the role of the spectral parameter of the reduced spectral problems.

- Five possible reductions arise from the Classical Symmetries. Three of them yield linear equations that can be easily integrated, providing us with nontrivial solutions .
- The other three reductions yield interesting Lax pairs of second, third and fourth order respectively. The introduction of the spectral parameters in the $1 + 1$ reductions arise in an absolutely natural way.

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