Quasi-Exact Solvability of Heun and related equations

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Outline

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Introduction

- The Heun Equation (HEq)

\[ u''(z) + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right) u'(z) + \frac{\alpha\beta z - q}{z(z-1)(z-a)} u(z) = 0 \]

is analyzed in the context of Quasi-Exactly Solvable systems associated with the Lie algebra \( sl(2, \mathbb{R}) \).

- Goals:

  - To prove that the Heun equation is QES under certain conditions of the parameters.
  - The four confluent cases: Confluent (CHEq), Bi-confluent (BCHEq), Double-confluent (DCHEq) and Tri-confluent (TCHEq) Heun equations inherit this property.
  - The four confluent cases are the canonical forms for QES spectral problems that admit normalizable solutions.

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Heun Equation

Heun equation is the Fuchsian equation with four singularities, i.e. the second-order linear differential equation defined in the Riemann sphere with four regular singular points.

It is written in canonical natural form, with the finite regular singular points located at $z = 0$, $z = 1$ and $z = a$, as

$$D_H u(z) = q u(z)$$

$$D_H = z(z - 1)(z - a) \frac{d^2}{dz^2} + (\gamma(z - 1)(z - a) + \delta z(z - a) + \epsilon z(z - 1)) \frac{d}{dz} + \alpha \beta z$$

$$\begin{pmatrix}
0 & 1 & a & \infty \\
0 & 0 & 0 & \alpha \\
1 - \gamma & 1 - \delta & 1 - \epsilon & \beta
\end{pmatrix}; \quad q$$

Fuchs condition:

$$\gamma + \delta + \epsilon = \alpha + \beta + 1$$
Confluent Forms of Heun Equation

**CHEq:** Confluence of \( z = a \) and \( z = \infty \) into an irregular singular point at \( z = \infty \) of rank-1 confluent type.

\[
D_{CH} u(z) = q u(z)
\]

\[
D_{CH} = z(z - 1) \frac{d^2}{dz^2} + (\gamma(z - 1) + \delta z + \epsilon z(z - 1)) \frac{d}{dz} + \alpha z
\]

**DCHeq:** Confluence of \( z = a \) with \( z = \infty \) and \( z = 1 \) with \( z = 0 \) into two irregular singular points at \( z = \infty \) and \( z = 0 \) of rank-1 confluent type.

\[
D_{DCHeq} u(z) = q u(z)
\]

\[
D_{DCHeq} = z^2 \frac{d^2}{dz^2} + \left( z^2 + \gamma z + \delta \right) \frac{d}{dz} + \alpha z
\]
Confluent Forms of Heun Equation

**BCHEq**: Confluence of \( z = 1, z = a \) and \( z = \infty \) into an irregular singular point at \( z = \infty \) of rank-2 confluent type.

\[
D_{BCH} u(z) = q u(z)
\]

\[
D_{BCH} = z \frac{d^2}{dz^2} + \left( \gamma + \delta z + z^2 \right) \frac{d}{dz} + \alpha z
\]

**TCHEq**: Confluence of \( z = 0 \) with the \( z = \infty \) point of the BCHeq all the original singular points reduce to an irregular singular point at \( z = \infty \) of rank-3 confluent type.

\[
D_{TCH} u(z) = q u(z)
\]

\[
D_{TCH} = \frac{d^2}{dz^2} + \left( \gamma + z^2 \right) \frac{d}{dz} + \alpha z
\]
Quasi-Exact Solvable Systems

Quasi-Exactly-Solvable (QES) systems are spectral problems characterized by the fact that part of the eigenvalues and eigenfunctions, but not the whole spectrum, can be found algebraically.

An important class of QES systems are characterized by a Hamiltonian that is an element of the enveloping algebra of a finite-dimensional Lie algebra of differential operators, which in turn admits a finite dimensional invariant module of smooth functions.

Quasi-Exact Solvable Systems

For a generic second-order (Lie-algebraic) QES spectral problem, \( \mathcal{H} f(z) = \lambda f(z) \), the Hamiltonian operator \( \mathcal{H} \) is written (possibly after an adequate “gauge" transformation) as a quadratic combination:

\[
\mathcal{H} = c_+ J^+ + c_0 J^0 + c_- J^- + c_{++} (J^+)^2 + c_{00} (J^0)^2 + c_{--} (J^-)^2 + \\
+ c_{+0} \left( J^+ J^0 + J^0 J^+ \right) + c_{+-} (J^+ J^- + J^- J^+) + c_{0-} \left( J^0 J^- + J^- J^0 \right)
\]

of \( J^- \), \( J^+ \) and \( J^0 \), the generators of the \( sl(2, \mathbb{R}) \) Lie algebra:

\[
J^- = \frac{d}{dz}, \quad J^0 = z \frac{d}{dz} - \frac{n}{2}, \quad J^+ = z^2 \frac{d}{dz} - nz.
\]

\[
\left[ J^0, J^+ \right] = J^+, \quad \left[ J^0, J^- \right] = -J^-, \quad \left[ J^+, J^- \right] = -2J^0.
\]

\( n \) is a non-negative integer that determines the dimension of the invariant module.
HEq as a QES system

\( D_H \) is a quadratic combination of \( J^+ \), \( J^- \) and \( J^0 \) if and only if one of the characteristic exponents associated to the singularity at infinity, i.e. \( \alpha \) or \( \beta \), is a non-positive integer.

This QES condition corresponds with a very well known result in the theory of Heun equation: \textit{Heun polynomials} exists if and only if \( \alpha \) or \( \beta \) belongs to \( \mathbb{Z}_- \).

Case \( \alpha = -n \):

\[
D_H = \left( \beta + \frac{n - 1}{2} \right) J^+ + \left( -a(\gamma + \delta + n - 1) - \beta + \delta \right) J^0 + \frac{a}{2} \left( 2\gamma + n - 1 \right) J^- + \\
- \frac{2(a(\gamma + \delta + n) + \beta - \delta + 1)}{n + 2} (J^0)^2 + \frac{1}{2} \left( J^+ J^0 + J^0 J^+ \right) \\
+ \frac{a(2(\gamma + \delta - 1) + n) + 2\beta - 2\delta - n}{2(n + 2)} (J^+ J^- + J^- J^+) + \frac{a}{2} \left( J^0 J^- + J^- J^0 \right)
\]
HEq as a QES system

The polynomial solutions are obtained in the standard way:

Considering the Frobenius solution at $z = 0$:

$$Q(z) = \sum_{k=0}^{\infty} \frac{P_k(q)}{k! a^k(\gamma)_k} z^k$$

the following three-term recurrence is verified:

$$P_{k+1} = (q + k(k - n + \beta - \delta + a(k + \gamma + \delta - 1)))P_k - a(k - n - 1)k(k + \beta - 1)(k + \gamma - 1)P_{k-1}$$

$Q(z)$ truncates to a polynomial of degree $n$ if $q$ is chosen as one of the $n + 1$ roots of $P_{n+1}(q)$.

$$Q_j(z) = \sum_{k=0}^{n} \frac{P_k(q_j)}{k! a^k(\gamma)_k} z^k, \quad j = 1, \ldots, n + 1$$

$$P_{n+1}(q) = \prod_{j=1}^{n+1} (q - q_j)$$
QES Solvability for the confluent equations

The different confluences in the Heun equation respect the QES property:

- **CHEq**: QES condition: $\alpha = -n \epsilon$.

$$D_{CH} = \epsilon J^+ + (\gamma + \delta + n - \epsilon - 1) J^0 - \frac{1}{2} (2\gamma + n - 1) J^- + \frac{2(\gamma + \delta + n - \epsilon)}{n + 2} (J^0)^2 \right.$$

$$\left. - \frac{2(\gamma + \delta - \epsilon - 1) + n}{2(n + 2)} (J^+ J^- + J^- J^+) - \frac{1}{2} \left( J^0 J^- + J^- J^0 \right) \right)$$

- **DCHEq**: QES condition: $\alpha = -n$.

$$D_{DCHE} = J^+ + (\gamma + n - 1) J^0 + \delta J^- + \frac{2(\gamma + n)}{n + 2} (J^0)^2 \right.$$

$$\left. - \frac{2\gamma + n - 2}{2(n + 2)} (J^+ J^- + J^- J^+) \right)$$
QES Solvability for the confluent equations

**BCHEq:** QES condition: $\alpha = -n$.

\[
D_{BCH} = J^+ + \delta J^0 - \left(\gamma + \frac{n-1}{2}\right) J^- + \frac{2\delta}{n+2} (J^0)^2
- \frac{\delta}{n+2} (J^+ J^- + J^- J^+) + \frac{1}{2} \left(J^0 J^- + J^- J^0\right)
\]

**TCHEq:** QES condition: $\alpha = -n$.

\[
D_{TCH} = J^+ + \gamma J^- + (J^0)^2
\]
Normalizable QES systems

Gonzalez-Kamran-Olver (1993) have classified QES spectral problems that admit normalizable wave functions.

Let $\mathcal{H}$ be a general quasi-exactly solvable operator:

$$-\mathcal{H} = P_2(z) \frac{d^2}{dz^2} + P_1(z) \frac{d}{dz} + P_0(z),$$

The spectral Schrödinger problem associated to $\mathcal{H}$ admits normalizable wave functions if the function $P_2(z)$ is of one of the following canonical forms:

1. $P_2(z) = (z^2 + 1), z \in (-\infty, \infty)$.
2. a) $P_2(z) = (z^2 - 1), z \in [1, \infty)$, b) $P_2(z) = (z^2 - 1), z \in (\infty, -1]$.
3. a) $P_2(z) = z^2, z \in (0, \infty)$, b) $P(z) = z^2, z \in (-\infty, 0)$.
4. $P_2(z) = z, z \in [0, \infty)$.
5. $P_2(z) = 1, z \in (-\infty, \infty)$.

Normalizable QES systems

Any quasi-exactly solvable operator of the general form

\[-H = P_2(z) \frac{d^2}{dz^2} + P_1(z) \frac{d}{dz} + P_0(z),\]

with \(P_2(z) > 0\) on an interval \(I \subset \mathbb{R}\), can be written in Schrödinger form:

\[\mu(z) \cdot H \cdot \frac{1}{\mu(z)} = -\frac{d^2}{dx^2} + V(x).\]

where \(x\) is determined by:

\[\left(\frac{dz}{dx}\right)^2 = P_2(z) \Rightarrow z = \zeta(x)\]

and \(\mu(z)\) is called the "gauge" factor.

If \(\varphi(z)\) is a wave function in the original coordinates, then \(\psi(x) = \mu(\zeta(x)) \varphi(\zeta(x))\) will be the corresponding wave function in the "Schrödinger" coordinates.
Normalizable QES systems and Confluent Heun Eqs.

- **CHEq.** The change of variable: \( z \to 2z - 1 \) moves the regular singular points in \( z = \pm 1 \). CHEq operator reads:

\[
D_{CH} = \left( z^2 - 1 \right) \frac{d^2}{dz^2} + \left( \frac{\epsilon}{2}(z^2 - 1) + \gamma(z - 1) + \delta(z + 1) \right) \frac{d}{dz} + \frac{\alpha}{2}(z + 1)
\]

and it belongs to the canonical Case 2 of normalizable canonical QES systems. Up to additive constants it is written in Schrödinger form as:

\[
-\mu(x) D_{CH} \frac{1}{\mu(x)} = -\frac{d^2}{dx^2} - a \cosh^2 x - b \cosh x - c \coth x \text{csch} x - d \text{csch}^2 x
\]

(1)

where \( x = \text{arccosh} z \), and \( \mu(z) = (z - 1)^{\frac{\gamma - 1}{4}} (z + 1)^{\frac{\delta - 1}{4}} e^{\frac{\epsilon z}{4}} \).

But general QES systems of type 2 that admit normalizable wave-functions have necessarily a potential term of the form (1).

Thus the Confluent Heun operator, in the QES case, represents the Canonical form of any QES spectral problem of type 2.
Normalizable QES systems and Confluent Heun Eqs.

• **DCHEq.** DCHEq belongs to the canonical Case 3 of normalizable canonical QES systems. Up to additive constants his Schrödinger form is:

\[
-\mu(x) D_{DCHEq} \frac{1}{\mu(x)} = -\frac{d^2}{dx^2} - a e^{-2x} - b e^{-x} - c e^x - d e^{2x}
\]

with \( x = \ln z \), and \( \mu(z) = z^{\frac{\gamma-1}{2}} e^{\frac{1}{2} \left( z - \frac{\delta}{z} \right)} \).

This expression corresponds with the general QES systems of type 3 that admit normalizable solutions.

• **BCHEq** and **TCHEq** ⇔ Cases 4 and 5.

What about Case 1?

It can be reduced to CHEq, after a Mœbius transformation that moves the regular singular points to \( z = \pm i \).
Summary

- Heun equation is a Quasi-Exactly Solvable system associated to the Lie Algebra $sl(2, \mathbb{R})$ if one of the characteristic exponents of $z = \infty$ is a non-positive integer.

- The processes that convert Heun Equation into the four confluent Heun equations preserves the QES character, leading to other four different QES systems associated to $sl(2, \mathbb{R})$.

- The four Confluent QES systems encompasses all the canonical cases of QES systems admitting normalizable wave-functions in the associated Schrödinger spectral problems.
Applications: 1) Demkov wave functions

"Demkov" wave functions are Elementary solutions to the quantum problem of diatomic molecular ions, in the Born-Oppenheimer approximation (i.e. the Two fixed Coulombian centers quantum problem)

- Eigenfunctions = irrational factors \( \times \) exponentials \( \times \) polynomials.
- Energy levels of "hydrogenoid" type.

The Schrödinger equation:

\[
\left[ -\frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) - \frac{Z_1}{r_1} - \frac{Z_2}{r_2} \right] \Psi = E \Psi
\]

is separable in spheroidal coordinates, i.e. the rotation about the focal axis of the 2D elliptic coordinates

\[
\xi = \frac{r_1 + r_2}{R} \in (1, +\infty), \quad \eta = \frac{r_2 - r_1}{R} \in (-1, 1)
\]

\[
x_1 = \frac{R}{2} \xi \eta, \quad x_2 = \frac{R}{2} \sqrt{\xi^2 - 1} \sqrt{1 - \eta^2} \cos \varphi, \quad x_3 = \frac{R}{2} \sqrt{\xi^2 - 1} \sqrt{1 - \eta^2} \sin \varphi
\]

- Demkov YN. 1968. "Elementary Solutions of the Quantum Problem of the Motion of a Particle in the Field of Two Coulomb Centers", JETP Lett. 7 76–79.
The search for separated solutions: \( \Psi(\xi, \eta, \varphi) = F(\xi) G(\eta) e^{im\varphi} \) lead to the "Radial" and "Angular" equations:

\[
\frac{d}{d\xi} \left[ (\xi^2 - 1) \frac{dF(\xi)}{d\xi} \right] + \left[ \frac{R^2 E}{2} (\xi^2 - 1) + R(Z_1 + Z_2) \xi + \lambda - \frac{m^2}{\xi^2 - 1} \right] F(\xi) = 0 \tag{2}
\]

\[
\frac{d}{d\eta} \left[ (1 - \eta^2) \frac{dG(\eta)}{d\eta} \right] + \left[ \frac{R^2 E}{2} (1 - \eta^2) + R(Z_2 - Z_1) \eta - \lambda - \frac{m^2}{1 - \eta^2} \right] G(\eta) = 0 \tag{3}
\]

with separation constants: \( m \in \mathbb{Z} \) and \( \lambda \in \mathbb{R} \).

Both equations are the **Generalized Spheroidal Equation** (GSEq), but with different intervals of definition: \( \xi \in (1, +\infty) \), \( \eta \in (-1, 1) \) and not identical parameters.

Two interesting sub-cases:

- \( Z_1 = Z_2 \Rightarrow \) Angular GSEq reduces to Spheroidal equation.
- \( Z_2 = 0 \Rightarrow \) Both equations are the same GSEq.

The Demkov approach consists in searching for eigen-functions corresponding to the energy levels of a hydrogenoid atom, i.e. \( E = -\frac{(Z_1+Z_2)^2}{2n_1^2} \), for the radial equation, and simultaneously in the angular one: \( E = -\frac{(Z_1-Z_2)^2}{2n_2^2} \), with \( n_1, n_2 \in \mathbb{N} \).
These two hypotheses are compatible only if the two energies coincide, i.e. there exist solutions only for values of $n_1$ and $n_2$ that solve the diophantine equation:

$$\frac{(Z_1 + Z_2)^2}{n_1^2} = \frac{(Z_1 - Z_2)^2}{n_2^2}$$

Moreover, the value of the separation constant $\lambda$ in the two equations obviously must be the same. The corresponding wave functions are "elementary":

$$\Psi(\xi, \eta, \varphi) = (\xi^2 - 1)^{\frac{|m|}{2}} (1 - \eta^2)^{\frac{|m|}{2}} u_r(\xi) u^a(\eta) e^{-\frac{R(Z_1+Z_2)}{2n_1}\xi} e^{-\frac{R(Z_1-Z_2)}{2n_2}\eta} e^{im\varphi}$$

where $u_r(\xi)$ and $u^a(\eta)$ are polynomials up to $n_1 - |m| - 1$ and $n_2 - |m| - 1$ order.

- There exists Demkov solutions only for several concrete charge distributions.
- Having fixed the charges, Demkov wave functions occur only for quantum numbers $n_1$ and $n_2$ that solve the diophantine equation.
- Forcing the same value of the separation constant $\lambda$ in the radial and angular equations is only possible for concrete values of the internuclear distance $R$. 
A similar situation can be achieved in the corresponding Planar problem:

\[
\left[ -\frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) - \frac{Z_1}{r_1} - \frac{Z_2}{r_2} \right] \Psi = E \Psi
\]

The equation is separable in 2D elliptic coordinates: \((\xi, \eta)\).

\[
(\xi^2 - 1) \frac{d^2 F(\xi)}{d\xi^2} + \xi \frac{dF(\xi)}{d\xi} + \left[ \frac{ER^2}{2} \xi^2 + R(Z_1 + Z_2)\xi + \lambda \right] F(\xi) = 0 \quad (4)
\]

\[
(1 - \eta^2) \frac{d^2 G(\eta)}{d\eta^2} - \eta \frac{dG(\eta)}{d\eta} - \left[ \frac{ER^2}{2} \eta^2 + R(Z_2 - Z_1)\eta + \lambda \right] G(\eta) = 0 \quad (5)
\]

(4) and (5) can be easily identified as the algebraic version of Razavy (REq) and Whittaker-Hill (WHEq) equations.

The existence of Demkov solutions in both the three and two dimensional cases is completely explained having into account that:

- GSEq, REq and WHEq are particular cases of CHEq,
- The hydrogenoid-type hypothesis represents exactly the QES condition for the corresponding CH equations.


Applications: 2) Stability of Kinks

Another interesting example is the Heun equation arising in the stability analysis of a kink solution in a $(1 + 1)$-dimensional field theory:

Let us consider the field theory in $1 + 1$ dimensions determined by the lagrangian density:

\[
\mathcal{L} = \frac{1}{2} \partial_\nu \phi \partial^\nu \phi - U(\phi) \quad ; \quad U(\phi) = \frac{\mu^2}{8(1 + \varepsilon^2)} \left( \phi^2 + \varepsilon^2 \right) \left( 1 - \phi^2 \right)^2
\]

The theory admits a well known kink solution:

\[
\phi_K(x) = \frac{\sinh \left( \frac{\mu x}{2} \right)}{\sqrt{1 + \frac{1}{\varepsilon^2} + \sinh^2 \left( \frac{\mu x}{2} \right)}}
\]

Graphical representation of $U(\phi)$ (left) and $\phi_K(x)$ (right), for $\mu = 1$, $\varepsilon^2 = \frac{1}{2}$.

And the stability analysis around \( \phi_K(x) \) leads to the Schrödinger equation:

\[
\left[ -\frac{d^2}{dx^2} + V(x) \right] \psi(x) = \lambda \psi(x) , \quad V(x) = \frac{d^2 U(\phi)}{d\phi^2} \bigg|_{\phi=\phi_K(x)}
\]

with:

\[
V(x) = \mu^2 \left( \sinh^4 \left( \frac{\mu x}{2} \right) - \frac{5 - \epsilon^2}{2\epsilon^2} \sinh^2 \left( \frac{\mu x}{2} \right) + \frac{(1 + \epsilon^2)(1 - 2\epsilon^2)}{4\epsilon^4} \right) \frac{1}{\left( 1 + \frac{1}{\epsilon^2} + \sinh^2 \left( \frac{\mu x}{2} \right) \right)^2}
\]

Translational invariance allows to calculate the ground state \((\lambda_0 = 0)\) in a simple way:

\[
\psi_0(x) \propto \frac{d\phi_K(x)}{dx}, \text{ i.e.}
\]

\[
\psi_0(x) \propto \mu \left( 1 + \frac{1}{\epsilon^2} \right) \cosh \left( \frac{\mu x}{2} \right) 2 \left( \left( 1 + \frac{1}{\epsilon^2} \right) + \sinh^2 \left( \frac{\mu x}{2} \right) \right)^{3/2} \equiv \left( 1 - \phi_K^2(x) \right) \sqrt{\phi_K^2(x) + \epsilon^2}
\]

Christ & Lee identify, for the particular case \( \epsilon^2 = \frac{1}{2} \), the second excited state:

\[
\lambda_2 = \frac{3}{4} \mu^2 ; \quad \psi_2(x) \propto \sqrt{1 - \phi_K^2(x)} \left( \phi_K^2(x) - \frac{1}{4} \right)
\]

Thus, for this value of \( \epsilon \), two eigen-states are known, but not the whole spectrum, we have a QES problem \(\Rightarrow\) is it a \(sl(2)\)-QES system?
Using as independent variable \( z = \phi_K(x)^2 \) the Schrödinger equation is written in algebraic form:

\[
\begin{align*}
&z(z - 1)^2(z + \varepsilon^2) \frac{d^2\psi}{dz^2} + (z - 1) \left( 2z^2 - \left( 1 - \frac{3\varepsilon^2}{2} \right) z - \frac{\varepsilon^2}{2} \right) \frac{d\psi}{dz} \\
&\quad - \frac{1}{4} (15z^2 - 6(2 - \varepsilon^2)z + 1 - 2\varepsilon^2)\psi(z) = \frac{1 + \varepsilon^2}{\mu^2} \lambda \psi(z) \tag{6}
\end{align*}
\]

Equation (6) is Fuchsian but it is not a canonical form of Heun equation.

- We know that Heun equation (in canonical natural form and for specific values of the characteristic exponents) is an \( sl(2) \)-QES system, but it is not true for equation (6).

- This fact has induced to several authors to present this example as a case of Quasi-Exactly solvability not associated to \( sl(2) \) and even to conjecture that in general Heun equation is not a \( sl(2) \) QES system.

The correct way to clarify this problem is obviously to convert equation (6) into a canonical natural Heun equation. The standard change of variable, determined by the characteristic exponents at \( z = 1 \) is:

\[
\psi(z) = (1 - z)^{1 - \frac{\lambda}{\mu^2}} f(z)
\]
\[ z(z - 1)(z + \varepsilon^2)f''(z) + \left( \gamma(z - 1)(z + \varepsilon^2) + \delta(z + \varepsilon^2) + \varepsilon(z - 1) \right)f'(z) + (\alpha \beta z - q)f(z) = 0 \quad (7) \]

\[ \gamma = \frac{1}{2}; \quad \delta = 1 + \sqrt{1 - \frac{\lambda}{\mu^2}}; \quad \epsilon = \frac{1}{2}; \quad \alpha = -\frac{3}{2} + \sqrt{1 - \frac{\lambda}{\mu^2}}; \quad \beta = \frac{5}{2} + \sqrt{1 - \frac{\lambda}{\mu^2}} \]

\[ q = (1 + \varepsilon^2) \frac{\lambda}{\mu^2} + \frac{\varepsilon^2}{2} \left( 1 - \sqrt{1 - \frac{\lambda}{\mu^2}} \right) - \frac{1}{4} \]

\[
\left\{ \begin{array}{cccc}
0 & 1 & -\varepsilon^2 & \infty \\
0 & 0 & 0 & -\frac{3}{2} + \sqrt{1 - \frac{\lambda}{\mu^2}} \\
\frac{1}{2} & -2\sqrt{1 - \frac{\lambda}{\mu^2}} & \frac{1}{2} & \frac{5}{2} + \sqrt{1 - \frac{\lambda}{\mu^2}}
\end{array} \right\} \]

Equation (7) is QES only if \( \alpha \) or \( \beta \) are a non-positive integer, but this impossible for \( \beta \) and only available for \( \alpha \) in the specific case:

\[ \lambda = \frac{3}{4} \mu^2 \Rightarrow \alpha = -1, \quad \beta = 3 \]

- Equation (7) is QES only for \( \lambda = \frac{3}{4} \mu^2 \), i.e. exactly the "magic" eigenvalue detected previously.

- We have a problem because to fix this value of \( \lambda \) automatically imposes that the spectral parameter \( q \) is necessarily: \( q = \frac{1}{2} + \varepsilon^2 \).
However we can calculate the set of two first-order polynomials that corresponds to the Heun equation with $\alpha = -n = -1$, considering a free valued spectral parameter $q$.

In this specific case the allowed values of $q$ are the two roots of $P_2(q)$:

$$ P_2(q) = q^2 - \frac{q}{4} - \frac{3}{4} \Rightarrow q_1 = 1, \quad q_2 = -\frac{3}{4} $$

and these values are compatible with the required $q = \frac{1}{2} + \varepsilon^2$ if and only if $\varepsilon^2 = \frac{1}{2} \Rightarrow q = 1 = q_1$, an thus it is explained the "magic" value of $\varepsilon^2$ previously obtained by trial and error methods.

Explicit calculation lead to the polynomial solution:

$$ f(z) = z - \frac{1}{4} $$

and inverting the changes of variables, the wave-function:

$$ \psi_2(x) \propto \sqrt{1 - \phi_K^2(x)} \left( \phi_K^2(x) - \frac{1}{4} \right), \quad \lambda = \frac{3}{4} \mu^2 $$
Graphical representation of $V(x)$ (left) for $\mu = 1$, $\varepsilon^2 = \frac{1}{2}$, and $\psi_0(x)$ (green) and $\psi_2(x)$ (blue).

In conclusion:

- The original Schrödinger equation is transformable into a Heun equation that is a QES system only for a concrete value of the Schrödinger eigenvalue.
- QES condition also fixes the spectral Heun parameter.
- Compatibility of these requirements determines that only one of the polynomial solutions of the Heun equation is associated to a genuine eigenfunction of the physical problem.
Thank you very much